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# The supersymmetric tensor hierarchy of $N = 1, d = 4$ supergravity

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Abstract: In this paper we construct the supersymmetric tensor hierarchy of N=1, d=4 supergravity. We find some differences with the general bosonic construction of 4-dimensional gauged supergravities.

The global symmetry group of $N = 1, d = 4$ supergravity consists of three factors: the scalar manifold isometry group, the invariance group of the complex vector kinetic matrix and the U(1) R-symmetry group. In contrast to (half)-maximal supergravities, the latter two symmetries are not embedded into the isometry group of the scalar manifold. We identify some components of the embedding tensor with Fayet-Iliopoulos terms and we find that supersymmetry implies that the inclusion of R-symmetry as a factor of the global symmetry group requires a non-trivial extension of the standard $p$-form hierarchy. This extension involves additional 3- and 4-forms. One additional 3-form is dual to the superpotential (seen as a deformation of the simplest theory).

We study the closure of the supersymmetry algebra on all the bosonic $p$-form fields of the hierarchy up to duality relations. In order to close the supersymmetry algebra without the use of duality relations one must construct the hierarchy in terms of supermultiplets. Such a construction requires fermionic duality relations among the hierarchy's fermions and these turn out to be local.

Keywords: Supersymmetry and Duality, Supergravity Models, Gauge Symmetry

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## Contents



## 1 Introduction

The embedding tensor formalism,[1] introduced in refs. [4–8] allows the study of the most general gaugings of field theories and, in particular, of supergravity theories. So far, it has been applied to maximally- and half-maximally-extended supergravities in various dimensions [9–14], but not (or, at least, as we are going to see, not in detail[2]) to supergravities with less supersymmetry, in particular $N = 1, 2$ in $d = 4$ and minimal supergravities in $d = 5, 6$.

[1] For recent reviews see refs. [1–3].

[2] The $N = 2, d = 4$ case has been partially studied in ref. [31].



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A crucial difference between these two cases is that in the former the global symmetries of the ungauged theories that act on the fermionic fields (the group of automorphisms of the supersymmetry algebra or R-symmetry) $H_{\rm aut}$ also act on the bosonic fields of the theory, while in the latter they do not. More precisely, if we denote by $G$ the global symmetry group of the ungauged theories and by $G_{\rm bos}$ the subgroup of $G$ that acts on the bosons, in the maximal or half-maximal supergravities, $H_{\rm aut} \subset G_{\rm bos} = G$. In particular, the scalars parametrize the coset $G/(H_{\rm aut} \times H_{\rm matter})$ where $H_{\rm matter}$ is related to the matter multiplets and it is trivial in maximally-extended supergravities.

The situation in $N = 1, 2$ supergravities in $d = 4$ or in minimal supergravities in $d = 5, 6$ is totally different: one can write $G = G_{\rm bos} \times H_{\rm aut}$. Further, in theories with low amounts of supersymmetry there may exist symmetries that act only on the vectors (and spinors) but not on the scalars. This is particularly clear in $N = 1, d = 4$ supergravity where one can take only vector supermultiplets and no chiral supermultiplets. The corresponding symmetry group is the invariance group of the complex vector kinetic matrix. These facts have to be taken into account properly and it is our goal to do so for the general case of $N = 1, d = 4$ supergravity by extending the recently found general 4-dimensional tensor hierarchy [15]. The 4-dimensional tensor hierarchy has also been studied in [16].

The tensor hierarchy [7, 8, 14–16]. is an interesting structure that arises as part of the embedding tensor formalism. It consists of a system of $p$-form potential fields of all degrees $p = 1, \cdots, d$ in terms of which one can construct gauge-covariant field strengths of all degrees $p = 2, \cdots, d$. The starting point in the construction of the tensor hierarchy associated to the gauging of some theory is the field content and global symmetry group of that theory. This global symmetry group is gauged using the embedding tensor formalism and in order to have gauge-covariant field strengths it is usually necessary to introduce higher-rank $p$-form potentials in a bootstrap procedure ending with the introduction of $p = d$-form potentials.

The extra $p$-form fields that one has to introduce in the construction of the hierarchy turn out to be dual to objects such as Noether currents, deformation parameters etc. of the field theory and, therefore, do not add new degrees of freedom when we add them to the theory. Only some of them are completely necessary to construct a gauge-invariant action for the gauged field theory. In the $d = 4$ case these are the 1- and 2-forms. Then, why should we be interested in the rest, apart from their need for consistency of the full construction?

Perhaps the main reason why one should be interested in all the higher-rank $p$-forms of a theory is the relation between supergravity $p$-form potentials and supersymmetric $(p-1)$-extended objects ("branes") which is at the core of many of the advances made over the last decade in String Theory. While the branes associated to higher-rank $p$-forms of the 10-dimensional supergravities are by now well known [22–25], little or nothing is known about those of supergravities with lower supersymmetry and dimensionality like $N = 1, 2$, $d = 4$ supergravity. For instance, in ref. [27] it was found that one can introduce, consistently with supersymmetry, 2-forms in $N = 2, d = 4$ supergravity associated with isometries of the scalar manifold to which strings couple. These 2-forms are "predicted" by the 4-dimensional tensor hierarchy [8]. But the 4-dimensional tensor hierarchy also predicts 3-forms and 4-forms and one would like to know if they can also be consistently



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introduced in the supergravity theory and the kind of extended objects (domain walls and spacetime-filling branes) they may couple to.

There is another reason to be interested in the higher-rank $p$-forms, in particular for $p = d-1$ of a supergravity theory. These $(d-1)$-form potentials are dual to the ("deformation") parameters that one can introduce consistently in the theory: gauge coupling constants (represented by the embedding tensor), Stückelberg masses etc. Finding all the $(d-1)$-form potentials one can get information about the most general deformations (gaugings, massive deformations...) of the theory.

In this paper we are going to study the possible $p$-form potentials that one can consistently add to $N = 1, d = 4$ supergravity, generalizing the results of ref. [19]. As we have explained, this problem is related to the construction of the tensor hierarchy associated to the most general (electric and magnetic) gauging of the theory. The global symmetry group $G$ of these theories can be written as[3] $G_{\rm iso} \times G_{\rm V} \times {\rm U}(1)_R$ where $G_{\rm iso}$ is the isometry group of the scalar manifold, $G_{\rm V}$ is the invariance group of the complex vector kinetic matrix and ${\rm U}(1)_R$ the R-symmetry group. We will use the general results of [15] but supersymmetry will force us to consider additional fields not contained in the standard tensor hierarchy. In particular, we will find a 3-form that can be interpreted as the dual to the superpotential which is a deformation of $N = 1, d = 4$ supergravity that is not related to a gauging. For earlier work on related questions see refs. [19, 23, 24, 26, 27].

This paper is organized as follows: in section 2 we review the standard electric gauging of perturbative symmetries of matter-coupled $N = 1, d = 4$ supergravity using the (electric part of the) embedding tensor. This will allow us to introduce our notation and conventions. In section 3 we introduce $N = 1, d = 4$ supergravity with electric and magnetic gaugings of perturbative and non-perturbative symmetries of the theory. This requires the use of the full embedding tensor and the introduction in the action of the 2-forms predicted by the general 4-dimensional tensor hierarchy. At this point we have a completely consistent theory with 1- and 2-forms and we do not need to introduce any higher-rank form potentials unless we worry about the gauge-covariant field strength of the 2-forms. This is necessary, though, to close (on-shell) the supersymmetry algebra on the 2-forms and we are led to consider all the $p$-form potentials predicted by the 4-dimensional tensor hierarchy. We, then, proceed to construct consistent (on-shell) supersymmetry transformations for all the hierarchy $p$-form potentials in section 3.3 which will lead us to extend the field content of the hierarchy. Finally, we review our results and present our conclusions in section 4. The appendices contain summaries of useful formulae concerning Kähler geometry and the 4-dimensional tensor hierarchy.

## 2 Electrically gauged $N = 1$, $d = 4$ supergravity

In this section we are going to describe the "standard" gauged $N = 1, d = 4$ theory [28] using the embedding-tensor formalism. By "standard" we mean that only perturbative global symmetries of the ungauged theory have been gauged using as gauge fields the

[3] This splitting of $G$ is factors is not unambiguous. In particular, ${\rm U}(1)_R$ transformations can be combined with transformations of the other two factors. We will discuss this in detail later.



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electric vector fields. In order to make as clear as possible the construction of the gauged theory, we are going to describe first the ungauged theory and its global symmetries and then the gauging procedure.

### 2.1 Ungauged $N = 1$, $d = 4$ supergravity

The basic[4] field content of any $N = 1, d = 4$ ungauged supergravity theory is a supergravity multiplet with one graviton $e^a{}_\mu$ and one chiral gravitino[5] $\psi_\mu$, $n_C$ chiral multiplets with as many chiralinos $\chi^i$ and complex scalars $Z^i$, $i = 1, \cdots, n_C$ that parametrize an arbitrary Kähler-Hodge manifold with metric $\mathcal{G}_{ij^*}$, and $n_V$ vector multiplets with as many Abelian vector fields $A^\Lambda$ with field strengths $F^\Lambda = dA^\Lambda$ and chiral gauginos $\lambda^\Lambda$, $\Lambda = 1, \cdots, n_V$.

In the ungauged theory the couplings between the above fields are determined by the Kähler metric[6] $\mathcal{G}_{ij^*}$, an arbitrary holomorphic kinetic matrix $f_{\Lambda\Sigma}(Z)$ with positive-definite imaginary part and an arbitrary holomorphic superpotential $W(Z)$ which appears through the covariantly holomorphic section of Kähler weight $(1, -1)$ $\mathcal{L}(Z, Z^*)$:

$$\mathcal{L}(Z, Z^*) = W(Z) e^{\mathcal{K}/2}\,, \tag{2.1}$$

so its Kähler-covariant derivative given in eq. (A.7) for $\bar{q} = -1$ is $\mathcal{D}_{i^*}\mathcal{L} = e^{\mathcal{K}/2}\partial_{i^*}W = 0$. In absence of scalar fields, it is possible to introduce a constant superpotential $\mathcal{L} = W = w$.

The chirality of the spinors is related to their Kähler weight: $\psi_\mu, \lambda^\Sigma$ and $\chi^i$ have the same chirality and $\psi_\mu, \lambda^\Sigma$ and $\chi^{*i^*}$ have the same Kähler weight $(1/2, -1/2)$ so their covariant derivatives take the form of eq. (A.9) with $q = 1/2$.

The action for the bosonic fields in the ungauged theory is

$$S_{\rm u} = \int \left[ \star R - 2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re{\rm e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_{\rm u} \right]\,, \tag{2.2}$$

where the scalar potential $V_{\rm u}$ is given by

$$V_{\rm u}(Z, Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*\,. \tag{2.3}$$

In absence of scalar fields the constant superpotential $\mathcal{L} = W = w$ leads to an anti-de Sitter-type cosmological constant

$$V_{\rm u} = -24|w|^2\,. \tag{2.4}$$

The supersymmetry transformation rules for the fermions (to first order in fermions) are

$$\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon + i\mathcal{L}\gamma_\mu \epsilon^* = \left[ \nabla_\mu + \frac{i}{2}\mathcal{Q}_\mu \right] \epsilon + i\mathcal{L}\gamma_\mu \epsilon^*\,, \tag{2.5}$$

$$\delta_\epsilon \lambda^\Lambda = \frac{1}{2}\not{F}^{\Lambda +}\epsilon\,, \tag{2.6}$$

$$\delta_\epsilon \chi^i = i\not{\partial} Z^i \epsilon^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\epsilon\,. \tag{2.7}$$

[4] In the ungauged classical theory (this work is only concerned with the classical theory) linear multiplets can always be dualized into chiral multiplets and so we do not need to deal with them. After the gauging, this is not possible in general, but the embedding tensor formalism will allow us to introduce the 2-forms in at a later stage in a consistent form.

[5] The conventions used here are essentially those of refs. [19] and [17].

[6] The elements of Kähler geometry needed in this paper are reviewed in appendix A.



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The last terms in eqs. (2.5) and 2.7) are *fermion shifts* associated to the superpotential which contribute quadratically to the potential $V_{\rm u}$.

In absence of scalar fields and with constant superpotential $\mathcal{L} = W = w$ the fermion shift in eq. (2.5) can be interpreted as part of an anti-de Sitter covariant derivative

$$\delta_\epsilon \psi_\mu = \nabla_\mu \epsilon + iw\gamma_\mu \epsilon^* \,. \tag{2.8}$$

The supersymmetry transformation rules for the bosonic fields (to the same order in fermions) are

$$\delta_\epsilon e^a{}_\mu = -\frac{i}{4}\bar\psi_\mu \gamma^a \epsilon^* + \text{c.c.}\,, \tag{2.9}$$

$$\delta_\epsilon A^\Lambda{}_\mu = \frac{i}{8}\bar\lambda^\Lambda \gamma_\mu \epsilon^* + \text{c.c.}\,, \tag{2.10}$$

$$\delta_\epsilon Z^i = \frac{1}{4}\bar\chi^i \epsilon \,. \tag{2.11}$$

### 2.2 Perturbative symmetries of the ungauged theory

The possible matter couplings of $N = 1, d = 4$ supergravities are quite unrestricted. As a result, the global symmetries of these theories can be very different from case to case. Depending on the couplings it is possible to have, at the same time, symmetry transformations that only act on certain fields and not on the rest and symmetry transformations that act simultaneously on all of them. Thus, it is not easy to describe all the possible global symmetry groups in a form that is at the same time unified and detailed without introducing a very complicated notation with several different kinds of indices. We are going to try to find an equilibrium between simplicity and usefulness.

Therefore, we are going to denote the group of all the global symmetries of the theory we work with[7] by $G$ and its generators by $T_A$ with $A, B, C = 1, \cdots, \text{rank}\, G$. They satisfy the Lie algebra

$$[T_A, T_B] = -f_{AB}{}^C T_C \,. \tag{2.12}$$

We denote by $G_{\rm bos}$ the subgroup of transformations of $G$ that act on the bosonic fields and its generators by $T_{\rm a}$ with $\mathrm{a}, \mathrm{b}, \mathrm{c} = 1, \cdots, \text{rank}\, G_{\rm bos} \leq \text{rank}\, G$. They satisfy the Lie subalgebra

$$[T_{\rm a}, T_{\rm b}] = -f_{\rm ab}{}^{\rm c} T_{\rm c} \,. \tag{2.13}$$

In $N = 1, d = 4$ supergravity we have $G = G_{\rm bos} \times \mathrm{U}(1)_R$ and $\text{rank}\, G_{\rm bos} = \text{rank}\, G - 1$. We split the indices accordingly as $A = (\mathrm{a}, \sharp)$. We may introduce a further splitting of the indices of $G_{\rm bos}$, $\mathrm{a} = (\mathbf{a}, \underline{\mathrm{a}})$ to distinguish between those that act on the scalars (holomorphic isometries, belonging to the group[8] $G_{\rm iso} \subset G_{\rm bos}$) and those that do not. The latter, as we

[7] In this section we will use this notation only for the perturbative symmetries and later on we will use the same notation for all symmetries. It should be easy to recognize from the context which case we are talking about.

[8] Not all the isometries of the metric will be perturbative or even non-perturbative symmetries of the full theory. They have to satisfy further conditions that we are going to study next. It is understood that, in order not to have a complicated notation, we denote by $G_{\rm iso}$ only those isometries which really are symmetries of the full theory and not the full group of isometries of $\mathcal{G}_{ij^*}$ (although they may eventually coincide).



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will see, constitute the subgroup $G_{\rm V} \subset G_{\rm bos}$ of symmetries that only act on the vector (super)fields and leave invariant the kinetic matrix $f_{\Lambda\Sigma}$. We have, then, $G_{\rm bos} = G_{\rm iso} \times G_{\rm V}$, since any bosonic symmetry transformation is either an element of $G_{\rm iso}$ or of $G_{\rm V}$ and further since by construction no element of $G_{\rm iso}$ can also be an element of $G_{\rm V}$ and vice versa.

Let us describe the $\mathrm{U}(1)_R$ transformations first. Under a $\mathrm{U}(1)_R$ transformation with constant parameter $\alpha^\sharp$, objects with Kähler weight $q$ are multiplied by the phase $e^{-iq\alpha^\sharp}$. All the fermions $\psi_\mu, \lambda^\Sigma, \chi^{*i^*}$, have a non-vanishing Kähler weight 1/2, though. All the bosons have zero Kähler weight and do not transform under $\mathrm{U}(1)_R$.

The superpotential $\mathcal{L}$ has a non-vanishing Kähler weight and therefore transforms under $\mathrm{U}(1)_R$. As a general rule, in the presence of a non-vanishing superpotential, $\mathrm{U}(1)_R$ will only be a symmetry of $N = 1, d = 4$ supergravity if the phase factor acquired by $\mathcal{L}$ in a $\mathrm{U}(1)_R$ transformation can be identified with a $\mathrm{U}(1)$ transformation of the scalars that leaves invariant the rest of the action. These transformations, which are necessarily isometries of the Kähler metric will be described next, but we can already give two examples to clarify the above statement.

1. Let us consider the case with no chiral superfields and, therefore, no scalars and a constant $\mathcal{L} = W = w$ giving rise to the potential eq. (2.4) and the gravitino supersymmetry transformation eq. (2.8). In this case $\mathrm{U}(1)_R$ transforms the complex constant $w$ into $e^{-i\alpha^\sharp} w$ and, therefore it is not a symmetry since symmetry transformations act on fields, not on coupling constants. Certainly, we can never gauge these transformations since the local phases would transform a constant into a function which is not a field.

2. Let us consider a theory with just one chiral supermultiplet, with Kähler potential $\mathcal{K} = |Z|^2$ and superpotential $W(Z) = wZ$ where $w$ is some complex constant so $\mathcal{L} = wZe^{|Z|^2/2}$. In this case $\mathrm{U}(1)_R$ transforms $\mathcal{L}(Z, Z^*)$ into $\mathcal{L}'(Z, Z^*) = we^{-i\alpha^\sharp} Ze^{|Z|^2/2}$. This transformation can be seen as a transformation of the scalar $Z' = e^{-i\alpha^\sharp} Z$ which happens to leave invariant the Kähler potential, metric etc. In this case $\mathrm{U}(1)_R$ is a symmetry when identified with a $\mathrm{U}(1)$ transformation acting on the complex scalar.

The $G_{\rm iso}$ transformations with constant parameters $\alpha^{\mathbf{a}}$ act on the complex scalars $Z^i$ as reparametrizations

$$\delta_\alpha Z^i = \alpha^{\mathbf{a}} k_{\mathbf{a}}{}^i(Z)\,. \tag{2.14}$$

If these transformations are symmetries of the full theory they must, first, preserve the metric $\mathcal{G}_{ij^*}$ and its Hermitean structure, which implies that the $k_{\mathbf{a}}{}^i$s are the holomorphic components of a set of Killing vectors $\{K_{\mathbf{a}} = k_{\mathbf{a}}{}^i \partial_i + k_{\mathbf{a}}^{*i^*} \partial_{i^*}\}$ that satisfy the Lie algebra of the group $G_{\rm iso}$

$$[K_{\mathbf{a}}, K_{\mathbf{b}}] = -f_{\mathbf{ab}}{}^{\mathbf{c}} K_{\mathbf{c}}\,. \tag{2.15}$$

The holomorphic and antiholomorphic components satisfy, separately, the same Lie algebra.

We can formally add to this algebra, vanishing "Killing vectors" $K_{\underline{\rm a}}$ associated to the transformations that do not act on the scalars (but do act on the vectors), so we have the full algebra of $G_{\rm bos}$

$$[K_{\rm a}, K_{\rm b}] = -f_{\rm ab}{}^{\rm c} K_{\rm c}\,. \tag{2.16}$$



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Further, we can also add another vanishing Killing vector $K_\sharp$, formally associated to $\mathrm{U}(1)_R$ and write the full Lie algebra of $G$

$$[K_A, K_B] = -f_{AB}{}^C K_C \,, \tag{2.17}$$

so the reparametrizations of the scalars $Z^i$ can be written as

$$\delta_\alpha Z^i = \alpha^A k_A{}^i(Z) \,. \tag{2.18}$$

The Killing property of the reparametrizations only ensures the invariance of the kinetic term for the scalars. In order to be symmetries of the full theory they must preserve the entire Kähler-Hodge structure and leave invariant the superpotential and the kinetic terms for the vector fields.

1. Let us start with the Kähler structure. The reparametrizations must leave the Kähler potential invariant up to Kähler transformations, i.e., for each Killing vector $K_A$

$$\pounds_A \mathcal{K} \equiv \pounds_{K_A} \mathcal{K} = k_A{}^i \partial_i \mathcal{K} + k_A^{*\,i^*} \partial_{i^*} \mathcal{K} = \lambda_A(Z) + \lambda_A^*(Z^*) \,. \tag{2.19}$$

This relation is consistent for $A = \underline{\mathrm{a}}, \sharp$, if

$$\Re\mathrm{e}\, \lambda_{\underline{\mathrm{a}}} = \Re\mathrm{e}\, \lambda_\sharp = 0 \,. \tag{2.20}$$

Furthermore, the reparametrizations must preserve the Kähler 2-form $\mathcal{J}$

$$\pounds_A \mathcal{J} = 0 \,. \tag{2.21}$$

The closedness of $\mathcal{J}$ implies that $\pounds_A \mathcal{J} = d(i_{K_A} \mathcal{J})$ and therefore the preservation of the Kähler structure implies the existence of a set of real functions $\mathcal{P}_A$ called *momentum maps* such that

$$i_{K_A} \mathcal{J} = d\mathcal{P}_A \,, \tag{2.22}$$

which is also consistent for $A = \underline{\mathrm{a}}, \sharp$ if the corresponding

$$\mathcal{P}_{\underline{\mathrm{a}}} = \mathcal{P}_\sharp = \text{ constant} \,. \tag{2.23}$$

Using only eq. (2.19) a local solution to eq. (2.22) is provided by

$$i\mathcal{P}_A = k_A{}^i \partial_i \mathcal{K} - \lambda_A \,, \tag{2.24}$$

which, on account of eq. (2.19) is equivalent to

$$i\mathcal{P}_A = -(k_A^{*\,i^*} \partial_{i^*} \mathcal{K} - \lambda_A^*) \,, \tag{2.25}$$

so that, for $A = \underline{\mathrm{a}}, \sharp$,

$$\lambda_{\underline{\mathrm{a}}} = -i\mathcal{P}_{\underline{\mathrm{a}}} \,, \qquad \lambda_\sharp = -i\mathcal{P}_\sharp \,, \tag{2.26}$$

where $\mathcal{P}_{\underline{\mathrm{a}}}$ and $\mathcal{P}_\sharp$ are real constants (see eq. (2.23)). Eq. (2.24) implies that the momentum maps can be used as prepotentials from which the Killing vectors can be derived:

$$k_{A\,i^*} = i\partial_{i^*} \mathcal{P}_A \,. \tag{2.27}$$



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Observe that this equation is consistent with the triviality of the "Killing vectors" $K_{\underline{a}}, K_\sharp$ and the constancy of the corresponding momentum maps eq. (2.23).

Using eqs. (2.17), (2.19) and (2.24) it can be shown that the momentum maps satisfy the so-called equivariance condition:

$$\pounds_A \mathcal{P}_B = 2ik_{[A}{}^i k^*_{B]}{}^{j^*} \mathcal{G}_{ij^*} = -f_{AB}{}^C \mathcal{P}_C \,. \tag{2.28}$$

This equivariance condition implies that momentum maps can only be constant and different from zero for Abelian factors. These constants will be associated after gauging to the *D*- or *Fayet-Iliopoulos* terms.

2. If the Kähler-Hodge structure is preserved, any section $\Phi$ of Kähler weight $(p, q)$ must transform as[9]

$$\delta_\alpha \Phi = -\alpha^A (\mathbb{L}_A - K_A)\Phi \,, \tag{2.29}$$

where $\mathbb{L}_A$ stands for the symplectic and Kähler-covariant Lie derivative w.r.t. $K_A$ and is given by

$$\mathbb{L}_A \Phi \equiv \left\{ \pounds_A + \left[ T_A + \frac{1}{2}(p\lambda_A + q\lambda^*_A) \right] \right\} \Phi \,, \tag{2.30}$$

where the $T_A$ are the matrices that generate the subgroup of $G_{\rm bos}$ that acts on the vectors. The $T_A$ are assumed to be in the representation in which the section transforms and they satisfy the Lie algebra eq. (2.12). This means that the gravitino $\psi_\mu$ transforms according to

$$\delta_\alpha \psi_\mu = -\frac{i}{2} \alpha^A \Im\mathrm{m}\, \lambda_A \psi_\mu \,. \tag{2.31}$$

For $A = \underline{a}, \sharp$ we have just $\mathrm{U}(1)_R$ transformations for each component $\mathcal{P}_{\underline{a}}, \mathcal{P}_\sharp$ different from zero. For $A = \mathbf{a}$ the transformations are still global but the $\Im\mathrm{m}\, \lambda_A$s are in general functions of $Z, Z^*$. These cannot be compensated by $\mathrm{U}(1)_R$ transformations.

The chiralinos $\chi^i$ transform according to

$$\delta_\alpha \chi^i = \alpha^A \left\{ \partial_j k_A{}^i \chi^j + \frac{i}{2} \Im\mathrm{m}\, \lambda_A \chi^i \right\} \,, \tag{2.32}$$

and the transformations of the gauginos will be discussed after we discuss the transformations of the vector fields.

3. Let us now consider the invariance of the superpotential $W$. We can require, equivalently, that the section $\mathcal{L}$ be invariant up to Kähler transformations. A Kähler-weight $(p, q)$ section $\Phi$ will be invariant up to Kähler transformations if[10]

$$\mathbb{L}_{\mathbf{a}} \Phi = 0 \,, \quad \Rightarrow \quad \pounds_{\mathbf{a}} \Phi = -\left[ T_{\mathbf{a}} + \frac{1}{2}(p\lambda_{\mathbf{a}} + q\lambda^*_{\mathbf{a}}) \right] \Phi \,. \tag{2.33}$$

[9] We do not write explicitly any spacetime, target space etc. indices.

[10] This condition only makes sense for transformations $K_{\mathbf{a}}$ that really act on the scalars.



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Therefore, we must require for all $A = \mathbf{a}$

$$K_{\mathbf{a}}\mathcal{L} = -i\Im\mathrm{m}\,\lambda_{\mathbf{a}}\mathcal{L}\,, \quad \Rightarrow \quad \delta_\alpha\mathcal{L} = -i\alpha^{\mathbf{a}}\Im\mathrm{m}\,\lambda_{\mathbf{a}}\mathcal{L}\,, \tag{2.34}$$

but we cannot extend straightforwardly the same expression to all $A$ since, as discussed at the beginning of this section, the corresponding transformations (constant phase multiplications) are only symmetries when $\mathcal{L} = 0$ or when they are associated to transformations of the scalars and this is, by definition, not the case when $A = \underline{\mathrm{a}}, \sharp$.

We, therefore, write

$$\delta_\alpha\mathcal{L} = -i\alpha^A\Im\mathrm{m}\,\lambda_A\mathcal{L}\,, \tag{2.35}$$

imposing at the same time the constraint[11]

$$(\alpha^{\underline{\mathrm{a}}}\Im\mathrm{m}\,\lambda_{\underline{\mathrm{a}}} + \alpha^\sharp\Im\mathrm{m}\,\lambda_\sharp)\mathcal{L} = (\alpha^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \alpha^\sharp\mathcal{P}_\sharp)\mathcal{L} = 0\,. \tag{2.36}$$

4. The kinetic term for the vector fields $A^\Lambda$ in the action will be invariant[12] if the effect of a reparametrization on the kinetic matrix $f_{\Lambda\Sigma}$ is equivalent to a rotation on its indices that can be compensated by a rotation of the vectors, or a constant Peccei-Quinn-type shift i.e.

$$\delta_\alpha f_{\Lambda\Sigma} \equiv -\alpha^{\mathrm{a}}\pounds_{\mathrm{a}} f_{\Lambda\Sigma} = \alpha^{\mathrm{a}}[T_{\mathrm{a}\,\Lambda\Sigma} - 2T_{\mathrm{a}\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega}]\,, \tag{2.37}$$

$$\delta_\alpha A^\Lambda = \alpha^{\mathrm{a}} T_{\mathrm{a}\,\Sigma}{}^{\Lambda} A^\Sigma\,, \tag{2.38}$$

where the shift generator is symmetric $T_{\mathrm{a}\,\Lambda\Sigma} = T_{\mathrm{a}\,\Sigma\Lambda}$ to preserve the symmetry of the kinetic matrix.

Observe that for $\mathrm{a} = \underline{\mathrm{a}}$, $\pounds_{\underline{\mathrm{a}}} f_{\Lambda\Sigma} = 0$, and, for consistency, we must have $T_{\underline{\mathrm{a}}\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} = 0$, i.e. the transformations $T_{\underline{\mathrm{a}}}$ are those that preserve the kinetic matrix. This is why we call the group generated by $T_{\underline{\mathrm{a}}}$ the invariance group $G_{\mathrm{V}}$ of the complex vector kinetic matrix.

The iteration of two of these infinitesimal transformations indicates that they can be described by the $2n_V \times 2n_V$ matrices[13]

$$T_{\mathrm{a}} \equiv \begin{pmatrix} T_{\mathrm{a}\,\Lambda}{}^{\Sigma} & 0 \\ \\ T_{\mathrm{a}\,\Lambda\Sigma} & T_{\mathrm{a}}{}^{\Lambda}{}_{\Sigma} \end{pmatrix}\,, \qquad T_{\mathrm{a}}{}^{\Lambda}{}_{\Sigma} \equiv -T_{\mathrm{a}\,\Sigma}{}^{\Lambda}\,, \tag{2.39}$$

satisfying the Lie algebra

$$[T_{\mathrm{a}}, T_{\mathrm{b}}] = -f_{\mathrm{ab}}{}^{\mathrm{c}} T_{\mathrm{c}}\,. \tag{2.40}$$

[11] This constraint should be understood as a way to consider the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously: when $\mathcal{L} \neq 0$ the symmetry transformations must satisfy $(\alpha^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \alpha^\sharp\mathcal{P}_\sharp) = 0$ and they are unrestricted when $\mathcal{L} = 0$.

[12] It is at this point that the restriction to perturbative symmetries (symmetries of the action) is made.

[13] Observe that this group is the semidirect product of the group that rotates the vectors, generated by the matrices $T_{\mathrm{a}\,\Sigma}{}^{\Lambda}$ and the Abelian group of shifts generated by the matrices $T_{\mathrm{a}\,\Lambda\Sigma}$. Evidently, some of these matrices identically vanish. This is the price we have to pay to use the same indices $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ for the generators of both groups.



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As we have discussed some of the transformations generated by the $K_{\mathbf{a}}$ may only act on the scalars and not on the vectors, for instance, because the kinetic matrix does not depend on the relevant scalars. We assume that the corresponding subset of $2n_V \times 2n_V$ matrices $T_{\mathbf{a}}$ are identically zero. On the other hand, we can formally add to these matrices another identically vanishing $2n_V \times 2n_V$ matrix $T_\sharp$ so we have a full set of $2n_V \times 2n_V$ matrices $T_A$ satisfying the Lie algebra of $G$, eq. (2.12).

Combining all these results we conclude that the gauginos transform according to

$$\delta_\alpha \lambda^\Sigma = -\alpha^A \left[ T_{A\,\Omega}{}^\Sigma \lambda^\Omega + \frac{i}{2} \Im\mathrm{m}\, \lambda_A \lambda^\Sigma \right] . \tag{2.41}$$

At this point there is no restriction on the group $G$ nor on the $n_V \times n_V$ matrices $T_{A\,\Lambda}{}^\Sigma$, although one can already see that the lower-triangular $2n_V \times 2n_V$ matrices $T_A$ are generators of the symplectic group.

### 2.3 Electric gaugings of perturbative symmetries

We are now going to gauge the symmetries described in the previous subsection using as gauge fields the electric 1-form potentials $A^\Lambda$. This requires the introduction of the (electric) *embedding tensor* $\vartheta_\Lambda{}^A$ to indicate which global symmetry is gauged by which gauge field $A^\Lambda$ and, equivalently, to identify the parameters of global symmetries $\alpha^A$ that are going to be promoted to local parameters with the gauge parameters $\Lambda^\Sigma(x)$ of the 1-forms:

$$\alpha^A(x) \equiv \Lambda^\Sigma(x) \vartheta_\Sigma{}^A . \tag{2.42}$$

We will write now the constraint eq. (2.36) in the form[14]

$$(\vartheta_\Sigma{}^{\underline{a}} \mathcal{P}_{\underline{a}} + \vartheta_\Sigma{}^\sharp \mathcal{P}_\sharp) \mathcal{L} = 0 . \tag{2.43}$$

Taking into account eq. (2.18) and the definition eq. (2.42), the gauge transformations of the complex scalars will be

$$\delta Z^i = \Lambda^\Sigma \vartheta_\Sigma{}^A k_A{}^i . \tag{2.44}$$

The embedding tensor cannot be completely arbitrary. To start with, it is clear that it has to be invariant under gauge transformations, which we denote by $\delta$:

$$\delta \vartheta_\Lambda{}^A = -\Lambda^\Sigma Q_{\Sigma\Lambda}{}^A , \qquad Q_{\Sigma\Lambda}{}^A \equiv \vartheta_\Sigma{}^B T_{B\,\Lambda}{}^\Omega \vartheta_\Omega{}^A - \vartheta_\Sigma{}^B \vartheta_\Lambda{}^C f_{BC}{}^A . \tag{2.45}$$

Then, the embedding tensor has to satisfy the quadratic constraint

$$Q_{\Sigma\Lambda}{}^A = 0 . \tag{2.46}$$

The gauge fields $A^\Lambda$ effectively couple to the generators

$$X_{\Sigma\Omega}{}^\Gamma \equiv \vartheta_\Sigma{}^A T_{A\,\Omega}{}^\Gamma , \qquad X_{\Sigma\Omega\Gamma} \equiv \vartheta_\Sigma{}^A T_{A\,\Omega\Gamma} , \qquad X_\Sigma \equiv \vartheta_\Sigma{}^A T_A . \tag{2.47}$$

[14] Again, this constraint and other constraints of the same kind that will follow, should be understood as a way to consider the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously: when $\mathcal{L} \neq 0$ the embedding tensor must satisfy $(\vartheta_\Sigma{}^{\underline{a}} \mathcal{P}_{\underline{a}} + \vartheta_\Sigma{}^\sharp \mathcal{P}_\sharp) = 0$ and it is unrestricted when $\mathcal{L} = 0$.



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From the definition of the quadratic constraint eq. (2.46)

$$X_{(\Lambda\Sigma)}{}^{\Omega}\vartheta_{\Omega}{}^{A} = 0\,, \tag{2.48}$$

which, for this purely electric gauging case implies

$$X_{(\Lambda\Sigma)}{}^{\Omega} = 0\,, \tag{2.49}$$

and no need to intoduce 2-form potentials. From the commutator of the matrices $T_A$ and using the quadratic constraint we find the commutator of $X$ generators

$$[X_\Lambda, X_\Sigma] = -X_{\Lambda\Sigma}{}^{\Omega}X_\Omega\,, \tag{2.50}$$

from which we can derive the analogue of the Jacobi identities.

We are now ready to gauge the theory. We will not attempt to give the full supersymmetric Lagrangian and supersymmetry transformation rules, but only those elements that allow its construction to lowest order in fermions (that is we consider supersymmetry transformations acting on fermions up to first order fermion terms and supersymmetry transformations acting on bosons up to second order fermions terms).

First, we have to replace the partial derivatives of the scalars in their kinetic term by the covariant derivatives

$$\mathfrak{D}Z^i \equiv dZ^i + A^\Lambda\vartheta_\Lambda{}^A k_A{}^i\,, \tag{2.51}$$

where the gauge potentials transform according to

$$\delta A^\Sigma = -\mathfrak{D}\Lambda^\Sigma \equiv -\left(d\Lambda^\Sigma + X_{\Lambda\Omega}{}^{\Sigma}A^\Lambda\Lambda^\Omega\right)\,. \tag{2.52}$$

We also replace in the action the vector field strengths by the gauge-covariant field strengths

$$F^\Sigma = dA^\Sigma + \frac{1}{2}X_{\Lambda\Omega}{}^{\Sigma}A^\Lambda\wedge A^\Omega\,. \tag{2.53}$$

Observe that we have not introduced a coupling constant $g$ as it is standard in the literature since the embedding tensor already plays the role of a coupling constant and even of different coupling constants if we are dealing with products of groups. Observe also that $\vartheta_\sharp{}^A$ does not appear in any of these expressions because $K_\sharp = T_\sharp = 0$.

We have to replace the (Kähler- and Lorentz-) covariant derivatives $\mathcal{D}$ of the spinors in their kinetic terms by the gauge-covariant derivatives $\mathfrak{D}$:

$$\mathfrak{D}_\mu\psi_\nu = \left\{\mathcal{D}_\mu - \frac{i}{2}A^\Lambda{}_\mu\vartheta_\Lambda{}^A\mathcal{P}_A\right\}\psi_\nu\,, \tag{2.54}$$

$$\mathfrak{D}\chi^i = \mathcal{D}\chi^i + \Gamma_{jk}{}^i\mathfrak{D}Z^j\chi^k - A^\Lambda\vartheta_\Lambda{}^A\partial_j k_A{}^i\chi^j + \frac{i}{2}A^\Lambda\vartheta_\Lambda{}^A\mathcal{P}_A\chi^i\,, \tag{2.55}$$

$$\mathfrak{D}\lambda^\Sigma = \left\{\mathcal{D} - \frac{i}{2}A^\Lambda\vartheta_\Lambda{}^A\mathcal{P}_A\right\}\lambda^\Sigma - X_{\Lambda\Omega}{}^{\Sigma}A^\Lambda\lambda^\Omega\,. \tag{2.56}$$

When $\mathcal{L} = 0$ the components $\vartheta_\Lambda{}^\sharp$ and $\vartheta_\Lambda{}^{\underline{a}}$ occur in all these covariant derivatives. When $\mathcal{L} \neq 0$ the embedding tensor $\vartheta_\Lambda{}^\sharp$ does not appear (and $\vartheta_\Lambda{}^{\underline{a}}$ only appears in the



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last term of $\mathfrak{D}\lambda^{\Sigma}$). In the case $\mathcal{L} \neq 0$ the gauging of the $\mathrm{U}(1)_R$ symmetry requires $\mathrm{U}(1)_R$ to be identified with a U(1) subgroup acting on the scalars. Thus the embedding tensor component associated to a $\mathrm{U}(1)_R$ gauging is contained in $\vartheta_{\Lambda}{}^{\mathbf{a}}$.

The supersymmetry transformations of the bosonic fields do not change with the gauging, but those of the fermions do by the above replacement of (Kähler- and Lorentz-) covariant derivatives by gauge-covariant derivatives. Further in the gaugino supersymmetry transformation the field strength is given by eq. (2.53) and there appears a new fermion shift term $\mathcal{D}^{\Sigma}$. To first order in fermions, we have

$$\delta_{\epsilon}\psi_{\mu} = \mathfrak{D}_{\mu}\epsilon + i\mathcal{L}\gamma_{\mu}\epsilon^{*}\,, \tag{2.57}$$

$$\delta_{\epsilon}\lambda^{\Sigma} = \frac{1}{2}\left[\not{F}^{\Sigma+} + i\mathcal{D}^{\Sigma}\right]\epsilon\,, \tag{2.58}$$

$$\delta_{\epsilon}\chi^{i} = i\,\not{\mathfrak{D}}Z^{i}\epsilon^{*} + 2\mathcal{G}^{ij^{*}}\mathcal{D}_{j^{*}}\mathcal{L}^{*}\epsilon\,, \tag{2.59}$$

where $\not{F}^{\Sigma+} = \gamma^{\mu}\gamma^{\nu}F^{\Sigma+}{}_{\mu\nu}$ in which $F^{\Sigma+} = \frac{1}{2}\left(F^{\Sigma} + i\star F^{\Sigma}\right)$ is the selfdual field strength, and

$$\mathcal{D}^{\Lambda} \equiv -\Im\mathrm{m}\, f^{\Lambda\Sigma}\vartheta_{\Sigma}{}^{A}\mathcal{P}_{A}\,, \tag{2.60}$$

where we use the notation

$$\Im\mathrm{m}\, f^{\Lambda\Sigma} \equiv (\Im\mathrm{m}\, f)^{-1|\Lambda\Sigma}\,. \tag{2.61}$$

The new term $\mathcal{D}^{\Lambda}$ leads to corrections of the scalar potential of the ungauged theory $V_{\mathrm{u}}$, given in eq. (2.3), which now takes the form

$$V_{\mathrm{eg}} = V_{\mathrm{u}} - \mathcal{D}^{\Lambda}\vartheta_{\Lambda}{}^{A}\mathcal{P}_{A} = V_{\mathrm{u}} + \frac{1}{2}\Im\mathrm{m}\, f^{\Lambda\Sigma}\vartheta_{\Lambda}{}^{A}\vartheta_{\Sigma}{}^{B}\mathcal{P}_{A}\mathcal{P}_{B}\,. \tag{2.62}$$

The action for the bosonic fields of the $N = 1, d = 4$ gauged supergravity of the kind considered here is obtained by replacing the partial derivatives and field strengths by gauge-covariant derivatives and field strengths, replacing the potential $V_{\mathrm{u}}$ by $V_{\mathrm{eg}}$ above and by adding a Chern-Simons term [29, 30] which is necessary to make the action gauge invariant

$$\begin{aligned} S_{\mathrm{eg}} = \int \Big\{ & \star R - 2\mathcal{G}_{ij^{*}}\mathfrak{D}Z^{i}\wedge\star\mathfrak{D}Z^{*\,j^{*}} - 2\Im\mathrm{m} f_{\Lambda\Sigma}F^{\Lambda}\wedge\star F^{\Sigma} + 2\Re\mathrm{e} f_{\Lambda\Sigma}F^{\Lambda}\wedge F^{\Sigma} \\ & - \star V_{\mathrm{eg}} - \frac{4}{3}X_{\Lambda\Sigma\Omega}A^{\Lambda}\wedge A^{\Sigma}\wedge\left[dA^{\Omega} + \frac{3}{8}X_{\Gamma\Delta}{}^{\Omega}A^{\Gamma}\wedge A^{\Delta}\right]\Big\}\,. \end{aligned} \tag{2.63}$$

Gauge-invariance can be achieved only if

$$X_{(\Lambda\Sigma\Omega)} = 0\,, \tag{2.64}$$

which is a constraint that also follows from supersymmetry.

## 3 Electrically and magnetically gauged $N = 1$, $d = 4$ supergravity

In this section we will discuss the most general gaugings of $N = 1, d = 4$ supergravity by using as gauge group any subgroup of $G = G_{\mathrm{iso}} \times G_{\mathrm{V}} \times \mathrm{U}(1)_R$ that can be embedded into $\mathrm{Sp}(2n_{\mathrm{V}}, \mathbb{R})$.



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From the purely bosonic point of view it would suffice to use the results of refs. [8, 15] taking into account the particular structure of the global symmetry group of $N = 1, d = 4$ supergravity. This involves the introduction of new $p$-form fields $p = 2, 3, 4$ which, together with the electric and magnetic (to be defined) 1-forms of the theory, combined into $A^M$, constitute the standard 4-dimensional tensor hierarchy, reviewed in appendices B and C. Its field content is

$$\{A^M, B_A, C_A{}^M, D_{AB}, D_E{}^{NP}, D^{NPQ}\}\,.$$

At the level of the action, it is not necessary to introduce all these fields, though. It is enough to introduce the magnetic 1-forms $A_\Lambda$ and 2-forms $B_A$.

This procedure, however, must be compatible with $N = 1, d = 4$ supersymmetry. A supersymmetrization of the tensor hierarchy and the action is necessary. The supersymmetrization of the tensor hierarchy is a first step towards the construction of a fully supersymmetric action with electric and magnetic gaugings and this is going to be our goal in this section.

Thus, we are going to repeat the construction of the 4-dimensional tensor hierarchy checking at each step its consistency with $N = 1, d = 4$ supersymmetry: for each new $p$-form field we will construct a supersymmetry transformation and we will check the closure of the local $N = 1, d = 4$ supersymmetry algebra on it. The commutator of two $N = 1, d = 4$ local supersymmetry transformations acting on bosonic $p$-form fields is expected to have the general form

$$[\delta_\eta\,, \delta_\epsilon] = \delta_{\text{g.c.t.}} + \delta_{\text{gauge}} + \text{duality relations}\,, \tag{3.1}$$

where $\delta_{\text{g.c.t.}}$ is a general coordinate transformation and $\delta_{\text{gauge}}$ is a gauge transformation that should coincide with the one predicted by the bosonic tensor hierarchy purely on the basis of gauge-invariance arguments. We also expect in general additional terms proportional to duality relations between the new fields and the original fields of the ungauged $N = 1, d = 4$ supergravity. These duality relations project the tensor hierarchy onto the physical theory reducing the number of independent fields.

Contrary to that expectation, we are going to see that, at least for some fields, it is possible to construct supersymmetry transformations such that the local $N = 1, d = 4$ supersymmetry algebra closes without the use of any duality relation, i.e.

$$[\delta_\eta\,, \delta_\epsilon] = \delta_{\text{g.c.t.}} + \delta_{\text{gauge}}\,. \tag{3.2}$$

To make this possible we will have to introduce the additional $p$-form fields of the tensor hierarchy in supermultiplets constructing, as a matter of fact, a supersymmetric tensor hierarchy. Now, to project the supersymmetric tensor hierarchy onto the physical theory we will use duality relations both for the bosons and fermions.

We have succeeded in supersymmetrizing in this way the hierarchy up to 2-forms (which requires the introduction of linear multiplets) but these results strongly indicate that the same should be possible for all $p$-forms in the tensor hierarchy.

Studying the closure of the local $N = 1, d = 4$ supersymmetry algebra we are going to see that it is necessary to add more bosonic $p$-form fields to the standard tensor hierarchy.



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The main reason for this is the existence of the constraint eq. (2.43) which will be generalized to the electric-magnetic case in eq. (3.30). This constraint restricts simultaneously the terms $\mathcal{P}_{\underline{a}}, \mathcal{P}_\sharp$ and the symmetries that can be gauged and reflects the breaking of the $\mathrm{U}(1)_R$ symmetry by the presence of a non-vanishing superpotential $\mathcal{L}$.

The breaking of this symmetry will manifest itself in the existence of a new Stückelberg shift of the 2-forms $B_{\underline{a}}, B_\sharp$

$$\delta B_{\underline{a}} \sim \mathcal{P}_{\underline{a}} \Lambda \,, \qquad \delta B_\sharp \sim \mathcal{P}_\sharp \Lambda \,, \tag{3.3}$$

where $\Lambda$ is a 2-form that appears whenever $\mathcal{L} \neq 0$. We can only find this shift by studying the closure of the local supersymmetry algebra. Therefore, it is necessary to simultaneously construct the tensor hierarchy and study its supersymmetrization.

To construct the respective gauge-covariant 3-form field strengths $H_{\underline{a}}$, $H_\sharp$ the existence of one new 3-form $C$ is required. We will find consistent supersymmetry transformations for the needed 3-form $C$ (as well as for yet another 3-form $C'$ that is dual to the superpotential). In order to have gauge-covariant 4-form field strengths $G_{\underline{a}}{}^M$ and $G_\sharp{}^M$ we need to introduce a set of 4-forms $D^M$. The extended hierarchy of $N = 1, d = 4$ supergravity will, thus, have the total bosonic field content

$$\{A^M, B_A, C_A{}^M, C, C', D_{AB}, D_E{}^{NP}, D^{NPQ}, D^M\} \,.$$

We start by reviewing the non-perturbative symmetries of the ungauged theory.

### 3.1 Non-perturbative symmetries of the ungauged theory

The new, non-perturbative symmetries to be considered are symmetries of the "extended" equations of motion of the ungauged theory which are the standard equations of motion plus the Bianchi identities of the vector field strengths:

$$dF^\Lambda = 0 \,. \tag{3.4}$$

The Maxwell equations that one obtains from the action eq. (2.2) can be written as Bianchi identities for the 2-forms $G_\Lambda$

$$dG_\Lambda = 0 \,, \qquad G_\Lambda{}^+ \equiv f_{\Lambda\Sigma}(Z) F^{\Sigma\,+} \,. \tag{3.5}$$

This set of extended equations of motion (Maxwell equations plus Bianchi identities) is invariant under general linear transformations

$$\begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}' = \begin{pmatrix} A_\Sigma{}^\Lambda & B^{\Sigma\Lambda} \\ C_{\Sigma\Lambda} & D^\Sigma{}_\Lambda \end{pmatrix} \begin{pmatrix} F^\Sigma \\ G_\Sigma \end{pmatrix} \,. \tag{3.6}$$

However, consistency with the definition of $G_\Lambda$ eq. (3.5) requires that the kinetic matrix transforms at the same time as

$$f' = (C + Df)(A + Bf)^{-1} \,. \tag{3.7}$$



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Then $f'$ will be symmetric if

$$A^T C - C^T A = 0\,, \qquad B^T D - D^T B = 0\,, \qquad A^T D - C^T B = \xi \mathbb{I}_{n_V \times n_V}\,, \tag{3.8}$$

where $\xi$ is a constant whose value is found to be $\xi = 1$ by the requirement of invariance of the Einstein equations.

These conditions can be reexpressed in a better form after introducing some notation. We define the contravariant tensor of 2-forms $G^M$, the symplectic metric $\Omega_{MN}$ and its inverse $\Omega^{MN}$ which we will use to, respectively, lower and raise indices

$$G^M \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}\,, \qquad \Omega_{MN} = \begin{pmatrix} 0 & \mathbb{I}_{n_V \times n_V} \\ -\mathbb{I}_{n_V \times n_V} & 0 \end{pmatrix}\,, \qquad \Omega^{MN}\Omega_{NP} = -\delta^M{}_P\,. \tag{3.9}$$

Then, the Maxwell equations and Bianchi identities are formally invariant under the transformations

$$G'^M \equiv M_N{}^M G^N\,, \qquad M = (M_N{}^M) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{3.10}$$

satisfying

$$M^T \Omega M = \Omega\,. \tag{3.11}$$

i.e. $M \in \mathrm{Sp}(2n_V, \mathbb{R})$ [20]. Infinitesimally[15]

$$M_N{}^M \sim \mathbb{I}_{2n_V \times 2n_V} + \alpha^A T_{A\,N}{}^M = \alpha^A \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix}\,, \tag{3.12}$$

and the condition $M \in \mathrm{Sp}(2n_V, \mathbb{R})$ reads

$$T_{A\,[MN]} \equiv T_{A\,[M}{}^P \Omega_{N]P} = 0\,. \tag{3.13}$$

These transformations change the kinetic matrix and will only be symmetries of all the extended equations of motion if they can be compensated by reparametrizations, i.e. $f_{\Lambda\Sigma}$ has to satisfy

$$\alpha^A k_A{}^i \partial_i f_{\Lambda\Sigma} = \alpha^A \left\{ -T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma} \right\}\,. \tag{3.14}$$

The subalgebra of matrices that generate symmetries of the action (perturbative symmetries) are those with $T_A{}^{\Sigma\Lambda} = 0$, i.e. the lower-triangular matrices of eq. (2.39).

Observe that the transformations acting on the vectors are constrained to belong to $\mathrm{Sp}(2n_V, \mathbb{R})$. That this is a constraint follows from the fact that the global symmetry group $G$ is in general not a subgroup of $\mathrm{Sp}(2n_V, \mathbb{R})$. We can thus only gauge those subgroups of $G$ that can be embedded in $\mathrm{Sp}(2n_V, \mathbb{R})$.

[15] We include identically vanishing generators associated to $\mathrm{U}(1)_R$ etc. On the other hand, it is clear that the index $A$ refers now to more symmetries than in the perturbative case.



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The transformation rule of the kinetic matrix $f_{\Lambda\Sigma} \equiv R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}$ eq. (3.7) can be alternatively expressed using the $\mathrm{Sp}(2n_V, \mathbb{R})$ matrix

$$(\mathcal{M}^{MN}) \equiv \begin{pmatrix} I^{\Lambda\Sigma} & I^{\Lambda\Omega} R_{\Omega\Sigma} \\ R_{\Lambda\Omega} I^{\Omega\Sigma} & I_{\Lambda\Sigma} + R_{\Lambda\Omega} I^{\Omega\Gamma} R_{\Gamma\Sigma} \end{pmatrix}, \qquad I^{\Lambda\Omega} I_{\Omega\Sigma} = \delta^{\Lambda}{}_{\Sigma}\,, \tag{3.15}$$

which transforms linearly

$$\mathcal{M}' = M \mathcal{M} M^T\,. \tag{3.16}$$

### 3.2 General gaugings of $N = 1$, $d = 4$ supergravity

We now want to consider the most general gauging of $N = 1, d = 4$ supergravity, using perturbative and non-perturbative global symmetries and using electric and magnetic vectors, to be introduced next. In the ungauged theory we can introduce $n_V$ 1-form potentials $A_\Lambda$ and their field strengths $F_\Lambda = dA_\Lambda$. The Maxwell equations can be replaced by the first-order duality relation

$$G_\Lambda = F_\Lambda\,, \tag{3.17}$$

since now the Bianchi identity $dF_\Lambda = 0$ implies the standard Maxwell equation $dG_\Lambda = 0$. The magnetic vectors $A_\Lambda$ will be introduced in the theory as auxiliary fields and we will study them from the supersymmetry point of view later on. The electric $A^\Lambda$ and magnetic $A_\Lambda$ vectors will be combined into a symplectic vector $A^M$

$$A^M \equiv \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \qquad A_M \equiv \Omega_{MN} A^N = (A_\Lambda\,, -A^\Lambda)\,, \qquad A^M = A_N \Omega^{NM}\,, \tag{3.18}$$

and used as the gauge fields of the symmetries described in the previous subsection.

In order to use all the 1-forms $A^M$ as gauge fields we need to add a magnetic component to the embedding tensor, which becomes a covariant symplectic vector

$$\vartheta_M{}^A \equiv (\vartheta^{\Lambda A}\,, \vartheta_\Lambda{}^A)\,, \tag{3.19}$$

where the index $A$ ranges over all the generators of $G = G_{\mathrm{bos}} \times \mathrm{U}(1)_R$, so we have now

$$\alpha^A(x) \equiv \Lambda^M(x) \vartheta_M{}^A\,, \tag{3.20}$$

and the gauge transformations of the complex scalars, for instance, take the form

$$\delta Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i\,. \tag{3.21}$$

The embedding tensor describes the embedding of the gauge group into the global symmetry group $G$. The part of the global symmetry group that cannot be embedded into $\mathrm{Sp}(2n_{\mathrm{V}}, \mathbb{R})$ is irrelevant for the purpose of gauging. There is thus no loss in generality to replace the global symmetry group by $\mathrm{Sp}(2n_{\mathrm{V}}, \mathbb{R})$. In this sense the embedding tensor provides an embedding of the gauge group into $\mathrm{Sp}(2n_{\mathrm{V}}, \mathbb{R})$. Besides the embedding into $\mathrm{Sp}(2n_V, \mathbb{R})$ there are further constraints that decrease the rank of the group that we can actually gauge.



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For instance, we must impose the constraint

$$Q^{AB} \equiv \frac{1}{4}\vartheta^{[A|M}\vartheta^{|B]}{}_{M} = 0\,, \quad \Rightarrow \quad \vartheta^{AM}\vartheta_{M}{}^{B} = 0\,, \tag{3.22}$$

which guarantees that the electric and magnetic gaugings are mutually local [8] and we can go to a theory with only purely electric gaugings by a symplectic transformation.

The embedding tensor must satisfy further conditions. We define the matrices

$$X_{MN}{}^{P} \equiv \vartheta_{M}{}^{A}T_{AN}{}^{P}\,, \tag{3.23}$$

which satisfy

$$X_{MNP} = X_{MPN}\,, \tag{3.24}$$

on account of eq. (3.13). Observe that the components $\vartheta_{M}{}^{\sharp}$ are not present in the $X_{MNP}$ tensors. Further, we impose the quadratic constraint[16]

$$Q_{NM}{}^{A} \equiv \vartheta_{N}{}^{A}T_{AM}{}^{P}\vartheta_{P}{}^{A} - \vartheta_{N}{}^{A}\vartheta_{M}{}^{B}f_{AB}{}^{A} = 0\,, \tag{3.25}$$

to ensure invariance of $\vartheta_{M}{}^{A}$ and the *representation constraint* [8]

$$L_{MNP} \equiv X_{(MNP)} = X_{(MN}{}^{Q}\Omega_{P)Q} = 0\,. \tag{3.26}$$

This constraint is required by gauge invariance and supersymmetry.[17] It implies eq. (2.64) and also

$$X_{(MN)P} = -\frac{1}{2}X_{PMN} \quad \Rightarrow X_{(MN)}{}^{P} = Z^{PA}T_{AMN}\,, \tag{3.27}$$

where we have defined

$$Z^{PA} \equiv -\frac{1}{2}\Omega^{NP}\vartheta_{N}{}^{A}\,. \tag{3.28}$$

This definition and that of the other projectors that appear in the 4-dimensional hierarchy are collected in appendix B. The tensor $Z^{PA}$ will be used to project in directions orthogonal to the embedding tensor since, due to the first quadratic constraint eq. (3.22),

$$Z^{MA}\vartheta_{M}{}^{B} = 0\,. \tag{3.29}$$

Finally, it should be clear that the constraint eq. (2.43) on the triple product of embedding tensor, momentum maps and superpotential should be generalized to

$$(\vartheta_{M}{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_{M}{}^{\sharp}\mathcal{P}_{\sharp})\mathcal{L} = 0\,. \tag{3.30}$$

Regarding the gauging of the $\mathrm{U}(1)_R$ symmetry group we have the following possibilities. If $\mathcal{L} = 0$ then the gauging shows up in the covariant derivatives of the fermions through terms containing $\mathcal{P}_{\sharp}$. The covariant derivatives acting on the scalars and vectors do not 'see' this gauging because $K_{\sharp} = T_{\sharp} = 0$. If we have a non-vanishing superpotential then it

[16] Observe that $\vartheta_{M}{}^{\sharp}$ does not occur in $Q_{NM}{}^{A}$ either.

[17] In ref. [21] it has been shown how this constraint gets modified in the presence of anomalies and the modifications can cancel exactly the lack of gauge invariance of the classical action.



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must be that $\vartheta_M{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp} = 0$ and in order to gauge the $\mathrm{U}(1)_R$ symmetry it must be identified with a U(1) subgroup of $G_{\rm bos}$.

We define gauge-covariant derivatives of objects transforming according to $\delta\phi = \Lambda^M\delta_M\phi$ by

$$\mathfrak{D}\phi = d\phi + A^M\delta_M\phi\,. \tag{3.31}$$

The gauge fields transform according to

$$\delta A^M = -\mathfrak{D}\Lambda^M + \Delta A^M = -\left(d\Lambda^M + X_{NP}{}^M A^N\Lambda^P\right) + \Delta A^M\,, \tag{3.32}$$

where $\Delta A^M$ is a piece that we can add to this gauge transformation if it satisfies

$$\vartheta_M{}^A\Delta A^M = 0\,. \tag{3.33}$$

The covariant derivatives of the scalars, gravitino and chiralinos read

$$\mathfrak{D}Z^i = dZ^i + A^M\vartheta_M{}^A k_A{}^i\,, \tag{3.34}$$

$$\mathfrak{D}_\mu\psi_\nu = \left\{\mathcal{D}_\mu - \frac{i}{2}A^M{}_\mu\vartheta_M{}^A\mathcal{P}_A\right\}\psi_\nu\,, \tag{3.35}$$

$$\mathfrak{D}\chi^i = \mathcal{D}\chi^i + \Gamma_{jk}{}^i\mathfrak{D}Z^j\chi^k - A^M\vartheta_M{}^A\partial_j k_a{}^i\chi^j + \frac{i}{2}A^M\vartheta_M{}^A\mathcal{P}_A\chi^i\,. \tag{3.36}$$

Observe that $\Delta A^M$ drops automatically from the gauge transformations of these expressions because $A^M$ always comes projected by $\vartheta_M{}^A$.

It is clear that we need to introduce auxiliary "magnetic gauginos" $\lambda_\Lambda$ in order to construct a symplectic vector of gauginos $\lambda^M$ whose covariant derivative is

$$\mathfrak{D}\lambda^M = \left\{\mathcal{D} - \frac{i}{2}A^N\vartheta_N{}^A\mathcal{P}_A\right\}\lambda^M - X_{NP}{}^M A^N\lambda^P\,. \tag{3.37}$$

The magnetic gauginos are the supersymmetric partners of the magnetic 1-forms. We will discuss their supersymmetry transformation rules later.

So far, to introduce the general 4-dimensional embedding-tensor formalism we have introduced magnetic 1-forms $A_\Lambda$ and gauginos $\lambda_\Lambda$. As discussed at the beginning of this section, we have to find supersymmetry transformations for them and check the closure of the local $N = 1, d = 4$ supersymmetry algebra.

### 3.3 The supersymmetric hierarchy

Before we deal with the supersymmetry transformations of the magnetic 1-forms that we have introduced, we take one step back and study the closure of the local $N = 1, d = 4$ supersymmetry algebra on the 0-forms.

#### 3.3.1 The scalars $Z^i$

Their supersymmetry transformations are given by eq. (2.11), which we rewrite here for convenience:

$$\delta_\epsilon Z^i = \frac{1}{4}\bar\chi^i\epsilon\,. \tag{3.38}$$



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At leading order in fermions,

$$\delta_\eta \delta_\epsilon Z^i = \frac{1}{4}\overline{(\delta_\eta \chi^i)}\epsilon\,, \tag{3.39}$$

and all we need is the supersymmetry transformation for $\chi^i$. This is given in eq. (2.59), which we also rewrite here

$$\delta_\eta \chi^i = i\,\not{\mathfrak{D}} Z^i \eta^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\eta\,, \tag{3.40}$$

where we have to take into account that the covariant derivative $\mathfrak{D}Z^i$ is now given by eq. (3.34). We get

$$[\delta_\eta\,,\,\delta_\epsilon]Z^i = \delta_{\text{g.c.t.}}Z^i + \delta_h Z^i\,, \tag{3.41}$$

where $\delta_{\text{g.c.t.}}Z^i$ is a g.c.t. with infinitesimal parameter $\xi^\mu$

$$\delta_{\text{g.c.t.}}Z^i = \pounds_\xi Z^i = +\xi^\mu \partial_\mu Z^i\,, \tag{3.42}$$

$$\xi^\mu \equiv \frac{i}{4}(\bar\epsilon\gamma^\mu\eta^* - \bar\eta\gamma^\mu\epsilon^*)\,, \tag{3.43}$$

and where $\delta_h Z^i$ is the gauge transformation eq. (3.21) with gauge parameter $\Lambda^M$

$$\delta Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i\,, \tag{3.44}$$

$$\Lambda^M \equiv \xi^\mu A^M{}_\mu\,. \tag{3.45}$$

This is just a small generalization of the standard result in which electric and magnetic gauge parameters appear. As expected, no duality relations are required to close the local supersymmetry algebra on the $Z^i$.

### 3.3.2 The 1-form fields $A^M$

As we have mentioned before, to define supersymmetry transformations for the magnetic vectors $A_\Lambda$ it is convenient to introduce simultaneously magnetic gauginos[18] $\lambda_\Lambda$. This is equivalent to introducing $n_V$ auxiliary vector supermultiplets. Symplectic covariance suggests that we can write the following supersymmetry transformation rules for the electric and magnetic 1-forms and gauginos:

$$\delta_\epsilon A^M{}_\mu = -\frac{i}{8}\bar\epsilon^*\gamma_\mu\lambda^M + \text{c.c.}\,, \tag{3.46}$$

$$\delta_\epsilon \lambda^M = \frac{1}{2}\left[\not{F}^{M+} + i\not{\mathcal{D}}^M\right]\epsilon\,, \tag{3.47}$$

where $F^M$ is the gauge-covariant 2-form field strength of $A^M$, to be defined shortly, and where we have defined the symplectic vector

$$\mathcal{D}^M \equiv \begin{pmatrix}\mathcal{D}^\Lambda \\ \mathcal{D}_\Lambda\end{pmatrix} \equiv \begin{pmatrix}\mathcal{D}^\Lambda \\ f_{\Lambda\Sigma}\mathcal{D}^\Sigma\end{pmatrix}\,, \tag{3.48}$$

[18] Magnetic gauginos have also been introduced in ref. [31].



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where now, the electric $\mathcal{D}^\Lambda$ has been redefined, with respect to the purely electric gauging case, to include a term with the magnetic component of the embedding tensor $\vartheta^{\Lambda A}$:

$$\mathcal{D}^\Lambda = -\Im\mathrm{m} f^{\Lambda\Sigma} (\vartheta_\Sigma{}^A + f^*_{\Sigma\Omega}\vartheta^{\Omega A})\mathcal{P}_A \,. \tag{3.49}$$

Although at this point we do not need it, it is important to observe that there is a duality relation between the magnetic gauginos and the electric ones

$$\lambda_\Lambda = f_{\Lambda\Sigma}\lambda^\Sigma \,. \tag{3.50}$$

The gaugino duality relation is local and takes the same form as the duality relation between the magnetic and the electric vector field strengths:

$$F_\Lambda{}^+ = f_{\Lambda\Sigma}F^{\Sigma\,+} \,, \tag{3.51}$$

which is obtained from the duality between electric and magnetic vectors $F_\Lambda = G_\Lambda$, combined with eq. (3.5). These duality relations relate the supersymmetry transformation $\delta_\epsilon\lambda^\Lambda$ to $\delta_\epsilon\lambda_\Lambda$.

Now we can check the closure of the local supersymmetry algebra on $A^M$. It is, however, convenient to know beforehand the form of the gauge transformations that we should expect on the right hand side of the commutator. The gauge transformations of $A^M$ are given in eq. (3.32) up to a term $\Delta A^M$ which is determined in the construction of the gauge-covariant field strength $F^M$. This term is also needed to have well-defined supersymmetry transformations for all the gauginos.

As shown in ref. [8], this requires the introduction of a set of 2-forms $B_A$ in $F^M$, which takes the form

$$F^M = dA^M + \frac{1}{2}X_{[NP]}{}^M A^N \wedge A^P + Z^{MA}B_A \,, \tag{3.52}$$

and is gauge-covariant under the transformations[19]

$$\delta_h A^M = -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A \,, \tag{3.53}$$

$$\delta_h B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}\left[\Lambda^N F^P + \frac{1}{2}A^N \wedge \delta_h A^P\right] + \Delta B_A \,, \qquad Z^{MA}\Delta B_A = 0 \,. \tag{3.54}$$

Let us now compute the commutator of two supersymmetry transformations on $A^M$. To leading order in fermions, eq. (3.46) gives

$$\delta_\eta\delta_\epsilon A^M{}_\mu = -\frac{i}{8}\bar\epsilon^*\gamma_\mu\delta_\eta\lambda^M + \text{c.c.} \tag{3.55}$$

Using eq. (3.47) with the parameter $\eta$, we find

$$[\delta_\eta\,,\,\delta_\epsilon]A^M{}_\mu = \xi^\nu F^M{}_{\nu\mu} + Z^{MA}\mathcal{P}_A\xi_\mu \,, \tag{3.56}$$

where $\xi^\mu$ is given by eq. (3.43) and we have used

$$\Im\mathrm{m}\mathcal{D}^M = 2Z^{MA}\mathcal{P}_A \,, \tag{3.57}$$

[19] The label $h$ in the gauge transformations indicates that these are the gauge transformations as predicted by the tensor hierarchy.



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which follows from the definitions eqs. (3.48), (3.49) and (B.1). We always expect a general coordinate transformation on the right hand side of the form

$$\delta_{\text{g.c.t.}} A^M{}_\mu = \pounds_\xi A^M{}_\mu = \xi^\nu \partial_\nu A^M{}_\mu + \partial_\mu \xi^\nu A^M{}_\nu \,. \tag{3.58}$$

Using the explicit form of the field strength $F^M$ eq. (3.52) we can rewrite it as

$$\delta_{\text{g.c.t.}} A^M{}_\mu = \xi^\nu F^M{}_{\nu\mu} + \mathfrak{D}_\mu (A^M{}_\nu \xi^\nu) + Z^{MA} \left[ B_{A\mu\nu} \xi^\nu - T_{A\,NP} A^N{}_\mu A^P{}_\nu \xi^\nu \right] . \tag{3.59}$$

Using this expression in the commutator and the definition eq. (3.45) of the gauge parameter $\Lambda^M$, we arrive at

$$[\delta_\eta \,,\, \delta_\epsilon] A^M = \delta_{\text{g.c.t.}} A^M + \delta_h A^M \,, \tag{3.60}$$

where, in complete agreement with the tensor hierarchy, $\delta_h A^M$ is the gauge transformation in eq. (3.53) with the 1-form gauge parameter $\Lambda_A$ given by

$$\Lambda_A \equiv -T_{A\,MN} A^N \Lambda^M + b_A - \mathcal{P}_A \xi \,, \tag{3.61}$$

$$b_{A\,\mu} \equiv B_{A\,\mu\nu} \xi^\nu \,. \tag{3.62}$$

Observe that no duality relation was needed to close the local supersymmetry algebra on the magnetic vector fields. This result is a consequence of using fully independent magnetic gauginos as the supersymmetric partners of the magnetic vector fields, i.e. transforming as $\delta_\epsilon \lambda_\Sigma \sim \not{F}_\Sigma{}^+$ instead of $\delta_\epsilon \lambda_\Sigma \sim \not{G}_\Sigma{}^+$. In the later case we would have gotten additional $G_\Sigma - F_\Sigma$ terms to be cancelled by using the duality relation.

### 3.3.3 The 2-form fields $B_A$

In order to have a gauge-covariant field strength $F^M$ for the 1-forms we have been forced to introduce a set of 2-forms $B_A$ and now we want to study the consistency of this addition to the theory from the point of view of supersymmetry and gauge invariance. We will first study the closure of the supersymmetry algebra on the 2-forms $B_A$ without introducing its supersymmetric partners and, later on, we will introduce the 2-forms as components of linear supermultiplets. In the first case, the local $N = 1, d = 4$ supersymmetry algebra will close up to the use of duality relations while in the second case it will close exactly.

It is useful to know beforehand what to expect on the right hand side of the commutator of two supersymmetry transformations acting on the 2-forms $B_A$. The gauge transformations of the 2-forms are given in eq. (3.54) up to a term $\Delta B_A$ which is constraint to satisfy $Z^{MA} \Delta B_A = 0$. In ref. ([15]) it was found that, in general,

$$\Delta B_A = -Y_{AM}{}^C \Lambda_C{}^M \,, \tag{3.63}$$

for some 2-form parameters $\Lambda_C{}^M$. $Y_{AM}{}^C$ is the projector given in eq. (B.2) and is annihilated by $Z^{NA}$ by virtue of the quadratic constraint eq. (2.46) (see eq. (B.6)), as required by the gauge-covariance of $F^M$. In generic 4-dimensional theories $Y_{AM}{}^C$ is the only tensor that is annihilated by $Z^{NA}$. At this point we have to remind ourselves that in $N = 1, d = 4$ supergravity there is another constraint, given in eq. (3.30), that may lead to



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additional terms in the gauge transformation of the 2-forms since eq. (3.30) can be written as $Z^{MA}(\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})\mathcal{L} = 0$. To see if there are any such additional terms in the gauge transformations of the 2-forms we need to compute the commutator of two supersymmetry transformations on $B_A$.

In any case, the generic tensor hierarchy prediction is that, with the gauge transformations eq. (C.2), which we rewrite here

$$\delta_h B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}\left[\Lambda^N F^P + \frac{1}{2}A^N \wedge \delta_h A^P\right] - Y_{AM}{}^C \Lambda_C{}^M\,, \tag{3.64}$$

the gauge-covariant field strength of $B_A$ is as given in eq. (C.8)

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge \left[dA^S + \frac{1}{3}X_{NP}{}^S A^N \wedge A^P\right] + Y_{AM}{}^C C_C{}^M\,, \tag{3.65}$$

where $C_C{}^M$ is a 3-form whose gauge transformations are determined to be

$$\delta_h C_C{}^M = \mathfrak{D}\Lambda_C{}^M - F^M \wedge \Lambda_C - \delta_h A^M \wedge B_C - \frac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta_h A^P + \Lambda^M H_C + \Delta C_C{}^M\,, \tag{3.66}$$

where

$$Y_{AM}{}^C \Delta C_C{}^M = 0\,. \tag{3.67}$$

We will next see that eq. (3.30) leads to additional terms in the 2-form gauge transformation. Inspired by the results of ref. [27], we found that, for the 2-forms $B_A$, the supersymmetry transformation is given by

$$\delta_\epsilon B_{A\mu\nu} = \frac{1}{4}[\partial_i \mathcal{P}_A \bar{\epsilon}\gamma_{\mu\nu}\chi^i + \text{c.c.}] + \frac{i}{2}[\mathcal{P}_A \bar{\epsilon}^* \gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{AMN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,. \tag{3.68}$$

The commutator of two of these supersymmetry transformations closes up to a duality relation to be described later on, a general coordinate transformation, and a gauge transformation of the form

$$\delta'_h B_A = \delta_h B_A - (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})\Lambda\,, \tag{3.69}$$

where $\delta_h B_A$ is the standard hierarchy gauge transformation eq. (C.2) and where the 2-form parameters $\Lambda$ and $\Lambda_C{}^M$ are given by

$$\Lambda_C{}^M \equiv -\Lambda^M B_C - c_C{}^M - \frac{1}{3}T_{CQP}\Lambda^P A^M \wedge A^Q\,, \tag{3.70}$$

$$\Lambda \equiv -c + 2\Re\text{e}(\phi\mathcal{L})\,, \tag{3.71}$$

$$\phi_{\mu\nu} \equiv \bar{\epsilon}^*\gamma_{\mu\nu}\eta^* = -\bar{\eta}^*\gamma_{\mu\nu}\epsilon^*\,, \tag{3.72}$$

$$c_C{}^M{}_{\mu\nu} \equiv C_C{}^M{}_{\mu\nu\rho}\xi^\rho\,, \tag{3.73}$$

$$c_{\mu\nu} \equiv C_{\mu\nu\rho}\xi^\rho\,. \tag{3.74}$$

The parameters $\Lambda^M$ and $\Lambda_A$ are, again, given by eqs. (3.45) and (3.61), respectively. We have introduced the anticipated 3-form $C$ with the gauge transformation

$$\delta'_h C = -d\Lambda\,, \tag{3.75}$$



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to take care of the Stückelberg shift parameter $\Lambda$. Strictly speaking we only need to introduce $C$ when $\mathcal{L} \neq 0$ in which case, according to the constraint eq. (3.30), $(\vartheta_M{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp}) = 0$. We can express this as a "constraint"

$$(\vartheta_M{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp})C = 0\,, \tag{3.76}$$

so

$$(\vartheta_M{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp})\Lambda = 0\,. \tag{3.77}$$

This constraint ensures that $Z^{MA}\Delta B_A = 0$ so that $F^M$ remains gauge-covariant under $\delta'_h B_A$.

The success of closing the supersymmetry algebra on the 2-forms, $B_A$, that is evaluating the commutator of two supersymmetry transformations (3.68), and showing that it gives rise to local symmetries acting on $B_A$ requires the use of the duality relation

$$H'_A = -\frac{1}{2} \star j_A\,, \tag{3.78}$$

where

$$j_A \equiv 2k^*_{A\,i}\mathfrak{D}Z^i + \text{c.c.}\,, \tag{3.79}$$

is the covariant Noether current 1-form and where the hierarchy gauge-covariant field strength $H_A$ given in eq. (C.8) has been modified to:

$$H'_A \equiv H_A - (\delta_A{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})C\,. \tag{3.80}$$

The modified field strength $H'_A$ transforms covariantly under the modified gauge transformations (3.69).

The right hand side of the duality relation (3.78) vanishes for $A = \underline{\mathrm{a}}, \sharp$. For these cases we expect to have currents bilinear in fermions which cannot appear at the order in fermions we are working at.

The origin of the extra term in eq. (3.80) that is proportional to $(\delta_A{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \delta_A{}^{\sharp}\mathcal{P}_{\sharp})$ can be traced back to the fact that the identity

$$\partial^{i^*}\mathcal{P}_{\mathbf{a}}\mathcal{D}_{i^*}\mathcal{L}^* - \mathcal{P}_{\mathbf{a}}\mathcal{L}^* = 0\,, \tag{3.81}$$

which is crucial for closing the supersymmetry algebra for the case $A = \mathbf{a}$ (it leads to a cancellation of terms coming from the supersymmetry variation of the first and second terms of eq. (3.68)) cannot be extended to the cases $A = \underline{\mathrm{a}}, \sharp$ in which we have introduced *fake* (vanishing) Killing vectors.

The introduction of the 3-forms $C$ and $C_A{}^M$ into the result for the commutator $[\delta_\eta, \delta_\epsilon]B_{A\mu\nu}$ via the duality relation (3.78) was necessary in order to make the result gauge invariant. Ultimately, this is only allowed if one can show that the supersymmetry algebra can also be closed on the 3-forms $C$ and $C_A{}^M$. This will be shown to be the case later on.



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### 3.3.4 The supermultiplet of $B_A$

We are now going to show that if we add to the tensor hierarchy full linear multiplets[20] $\{B_{A\,\mu\nu}, \varphi_A, \zeta_A\}$ where $\varphi_A$ is a real scalar and $\zeta_A$ is a Weyl spinor, instead of just the 2-forms $B_A$, as in the preceding section, we can close the local $N = 1, d = 4$ supersymmetry algebra on the 2-forms exactly without the use of the duality relation eq. (3.78).

We will construct the supersymmetry rules of the linear supermultiplet first for the case $A = \mathbf{a}$ after which this result will be generalized to include also the cases $A = \underline{a}, \sharp$. The above supersymmetry transformation rule eq. (3.68) suggests the fermionic duality rule

$$\zeta_{\mathbf{a}} = \partial_i \mathcal{P}_{\mathbf{a}} \chi^i = i k^*_{\mathbf{a}\,i} \chi^i \,, \tag{3.82}$$

so we would have

$$\delta_\epsilon B_{\mathbf{a}\mu\nu} = \frac{1}{4}[\bar\epsilon \gamma_{\mu\nu} \zeta_{\mathbf{a}} + \text{c.c.}] + \frac{i}{2}[\mathcal{P}_{\mathbf{a}} \bar\epsilon^* \gamma_{[\mu} \psi_{\nu]} - \text{c.c.}] + 2 T_{\mathbf{a}\,MN} A^M{}_{[\mu} \delta_\epsilon A^N{}_{\nu]} \,. \tag{3.83}$$

The supersymmetry transformation rule of $\zeta_{\mathbf{a}}$ follows from the above duality rule:

$$\delta_\epsilon \zeta_{\mathbf{a}} = i k^*_{\mathbf{a}\,i} \delta_\epsilon \chi^i = -k^*_{\mathbf{a}\,i} \not{\mathfrak{D}} Z^i \epsilon^* + 2 \partial_i \mathcal{P}_{\mathbf{a}} \mathcal{G}^{ij^*} \mathcal{D}_{j^*} \mathcal{L}^* \epsilon \,. \tag{3.84}$$

Using next the duality rule eq. (3.78) $j_{\mathbf{a}} = 4 \Re\text{e}(k^*_{\mathbf{a}\,i} \mathfrak{D} Z^i) = -2 \star H_{\mathbf{a}}$ we find

$$\delta_\epsilon \zeta_{\mathbf{a}} = -i \left[ \frac{i}{12} \not{H}_{\mathbf{a}} + \Im\text{m}(k^*_{\mathbf{a}\,i} \mathfrak{D}_\mu Z^i) \gamma^\mu \right] \epsilon^* + 2 \mathcal{P}_{\mathbf{a}} \mathcal{L}^* \epsilon \,. \tag{3.85}$$

To make contact with the standard linear multiplet supersymmetry transformations we should be able to identify consistently

$$\Im\text{m}(k^*_{\mathbf{a}\,i} \mathfrak{D} Z^i) \equiv \mathfrak{D} \varphi_{\mathbf{a}} \,, \tag{3.86}$$

for some real scalar $\varphi_{\mathbf{a}}$. The integrability condition of this equation can be obtained by acting with $\mathfrak{D}$ on both sides. Using on the l.h.s. the property

$$\mathfrak{D} k^*_{\mathbf{a}\,i} = \mathfrak{D} Z^{*j^*} \nabla_{j^*} k^*_{\mathbf{a}\,i} \,, \tag{3.87}$$

and the Killing property, the integrability condition takes the form

$$-i F^M \vartheta_M{}^{\mathbf{b}} k^*_{[\mathbf{a}|i} k_{|\mathbf{b}]}{}^i = f_{\mathbf{ab}}{}^{\mathbf{c}} F^M \vartheta_M{}^{\mathbf{b}} \varphi_{\mathbf{c}} \,, \tag{3.88}$$

which is solved by

$$-i k^*_{[\mathbf{a}|i} k_{|\mathbf{b}]}{}^i = f_{\mathbf{ab}}{}^{\mathbf{c}} \varphi_{\mathbf{c}} \,. \tag{3.89}$$

Given that the Killing vectors can be derived from the Killing prepotential $\mathcal{P}_{\mathbf{a}}$ which is equivariant, it follows that

$$k^*_{[\mathbf{a}|i} k_{|\mathbf{b}]}{}^i = \frac{i}{2} \pounds_{\mathbf{a}} \mathcal{P}_{\mathbf{b}} = -\frac{i}{2} f_{\mathbf{ab}}{}^{\mathbf{c}} \mathcal{P}_{\mathbf{c}} \,, \tag{3.90}$$

[20] Similar supermultiplets have been introduced in electro-magnetically gauged globally supersymmetric $N = 2, d = 4$ field theory [31].



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and we can finally identify

$$\Im\mathrm{m}(k^*_{\mathbf{a}\,i}\mathfrak{D}Z^i) = -\frac{1}{2}\mathfrak{D}\mathcal{P}_{\mathbf{a}}\,. \tag{3.91}$$

The supersymmetry transformations of the linear multiplet $\{B_{\mathbf{a}\,\mu\nu}, \varphi_{\mathbf{a}}, \zeta_{\mathbf{a}}\}$ are given by

$$\delta_\epsilon\zeta_{\mathbf{a}} = -i\left[\frac{1}{12}\not{H}_{\mathbf{a}} + \not{\mathfrak{D}}\varphi_{\mathbf{a}}\right]\epsilon^* - 4\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, \tag{3.92}$$

$$\delta_\epsilon B_{\mathbf{a}\mu\nu} = \frac{1}{4}[\bar\epsilon\gamma_{\mu\nu}\zeta_{\mathbf{a}} + \text{c.c.}] - i[\varphi_{\mathbf{a}}\bar\epsilon^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{\mathbf{a}\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, \tag{3.93}$$

$$\delta_\epsilon\varphi_{\mathbf{a}} = -\frac{1}{8}\bar\zeta_{\mathbf{a}}\epsilon + \text{c.c.}\,. \tag{3.94}$$

The duality relations needed to relate these fields to the fundamental fields of the $N = 1, d = 4$ gauged supergravity are

$$\zeta_{\mathbf{a}} = \partial_i\mathcal{P}_{\mathbf{a}}\chi^i\,, \tag{3.95}$$

$$H_{\mathbf{a}} = -\frac{1}{2}\star j_{\mathbf{a}}\,, \tag{3.96}$$

$$\varphi_{\mathbf{a}} = -\frac{1}{2}\mathcal{P}_{\mathbf{a}}\,. \tag{3.97}$$

The supersymmetry algebra closes on all the fields of the linear multiplet without the use of any duality relation.

Now that we know the supersymmetry transformation rules for $A = \mathbf{a}$ we will generalize them to all values of $A$. The supersymmetry transformations of the linear multiplet $\{B_{A\,\mu\nu}, \varphi_A, \zeta_A\}$ are given by

$$\delta_\epsilon\zeta_A = -i\left[\frac{1}{12}\not{H}'_A + \not{\mathfrak{D}}\varphi_A\right]\epsilon^* - 4\delta_A{}^{\mathbf{a}}\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, \tag{3.98}$$

$$\delta_\epsilon B_{A\mu\nu} = \frac{1}{4}[\bar\epsilon\gamma_{\mu\nu}\zeta_A + \text{c.c.}] - i[\varphi_A\bar\epsilon^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{A\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, \tag{3.99}$$

$$\delta_\epsilon\varphi_A = -\frac{1}{8}\bar\zeta_A\epsilon + \text{c.c.}\,. \tag{3.100}$$

The duality relations, eqs. (3.95) to (3.97), become

$$\zeta_A = \partial_i\mathcal{P}_A\chi^i\,, \tag{3.101}$$

$$H'_A = -\frac{1}{2}\star j_A\,, \tag{3.102}$$

$$\varphi_A = -\frac{1}{2}\mathcal{P}_A\,. \tag{3.103}$$

Observe that some terms on the right hand side are zero for $A = \underline{\mathrm{a}}, \sharp$, at least to leading order in fermions.

Now the gauge parameters that appear on the right hand side of the commutator of two supersymmetry transformations are different from those we found in the previous section and, therefore, do not match with those we found in the case of the 1-forms. To relate the parameters of the supersymmetry algebra in the case with and without the linear supermultiplets we also need to use the above duality relations. For instance, $\Lambda_A$ is given



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by eq. (3.61) with $\mathcal{P}_A$ replaced by $-2\varphi_A$. This means that, in order to supersymmetrize consistently the tensor hierarchy we also must replace $\mathcal{P}_A$ by $-2\varphi_A$ in the supersymmetry transformation rules of the gauginos eq. (3.47) (i.e. in the definition of $\mathcal{D}^M$ eqs. (3.48) and (3.49)). There are furthermore also 3-forms contained in the transformation rule for $\zeta_A$. Thus, if we continue this program we need to find a way to close the algebra on all the 3-forms without using any duality relations.

However, we will not pursue here any further the supersymmetrization of the tensor hierarchy for the higher-rank $p$-forms but we think that the above results strongly suggest that an extension with additional fermionic and bosonic fields of the tensor hierarchy on which the local supersymmetry algebra closes without the use of duality relations must exist. The duality relations must project the supersymmetric tensor hierarchy on to the $N = 1$ supersymmetric generalization of the (bosonic) action which will be given later in eq. (3.126).

As we have seen in the vector and 2-form cases, the duality relations among the additional fields (fermionic $\lambda_\Sigma, \zeta^A$ and bosonic $\varphi_A$) are local as opposed to those involving the original bosonic fields ($A_\Lambda$, $B_A$), which are non-local and related via Hodge-duality.

### 3.3.5 The 3-form fields $C_A{}^M$

We will be brief here because the construction of the field strength and the determination of the gauge transformations of the 3-forms $C_A{}^M$ are similar to those of the other fields.

We first remark that, in order to make the standard hierarchy's field strength $G_C{}^M$ gauge-invariant under the new gauge transformations, we must modify it as follows:

$$G'_A{}^M \equiv G_A{}^M + (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^\sharp\mathcal{P}_\sharp)D^M\,, \tag{3.104}$$

where $G_A{}^M$ is given in eq. (C.9) and $D^M$ is a 4-form transforming as

$$\delta'_h D^M = \mathfrak{D}\Sigma^M + \left(F^M - \frac{1}{2}Z^{MA}B_A\right)\wedge\Lambda\,, \tag{3.105}$$

and where we must also modify the gauge transformation rules of the 3-forms $C_A{}^M$ to be

$$\delta'_h C_A{}^M = \delta_h C_A{}^M - (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^\sharp\mathcal{P}_\sharp)\mathfrak{D}\Sigma^M\,. \tag{3.106}$$

In order to prove this result we have made use of the constraint eq. (3.30) and also of the fact, mentioned in section 2.2, that the directions $A = \underline{a}$ for which $\mathcal{P}_{\underline{a}} \neq 0$ must necessarily be Abelian, so

$$Y_{AM}{}^A(\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^\sharp\mathcal{P}_\sharp)\mathcal{L} = 0\,, \tag{3.107}$$

etc.

Then, the supersymmetry transformations of the 3-forms $C_A{}^M$ are given by

$$\delta_\epsilon C_A{}^M{}_{\mu\nu\rho} = -\frac{i}{8}\left[\mathcal{P}_A\bar{\epsilon}^*\gamma_{\mu\nu\rho}\lambda^M - \text{c.c.}\right] - 3B_{A\,[\mu\nu|}\delta_\epsilon A^M{}_{|\rho]} - 2T_{A\,PQ}A^M{}_{[\mu}A^P{}_{\nu|}\delta_\epsilon A^Q{}_{|\rho]}\,. \tag{3.108}$$

The local $N = 1, d = 4$ supersymmetry algebra closes on $C_A{}^M$ upon the use of a duality relation to be discussed later. The gauge transformations of $C_A{}^M$ that appear on



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the right hand side are the ones described above with

$$\Lambda_{BC} = d_{BC} + B_{[B} \wedge b_{C]} + 2T_{[B|\,NP}\Lambda^P A^N \wedge B_{C]}\,, \tag{3.109}$$

$$\Lambda^{NPQ} = d^{NPQ} + 2\Lambda^{(P} A^N \wedge (F^{Q)} - Z^{Q)C} B_C) - \frac{1}{4} X_{RS}{}^{(Q} \Lambda^P A^{N)} \wedge A^R \wedge A^S\,, \tag{3.110}$$

$$\Lambda_E{}^{NP} = d_E{}^{NP} - \Lambda^N C_E{}^P + \frac{1}{2} T_{E\,QR} \Lambda^Q A^N \wedge A^R \wedge A^P\,, \tag{3.111}$$

where $d_{BC\mu\nu\rho} = D_{BC\mu\nu\rho\sigma}\xi^\sigma$, and similarly for $d^{NPQ}$ and $d_E{}^{NP}$. The gauge transformation parameters $\Lambda^M$, $\Lambda_{\mathbf{a}}$ and $\Lambda_{\mathbf{a}}{}^M$ are, again, given by eqs. (3.45), (3.61) and (3.70), respectively.

In the closure of the local supersymmetry algebra we have made use of the duality relation

$$G'_A{}^M = -\frac{1}{2} \star \Re\mathrm{e}(\mathcal{P}_\mathrm{A} \mathcal{D}^\mathrm{M})\,. \tag{3.112}$$

According to the results of ref. [15], the duality relation has the general form

$$G'_A{}^M = \frac{1}{2} \star \frac{\partial V}{\partial \vartheta_M{}^A}\,. \tag{3.113}$$

Comparing these two expressions and using the relation between the potential of the supergravity theory and the fermion shifts, we conclude that, after the general electric-magnetic gauging the potential of $N = 1, d = 4$ supergravity is given by

$$V_{\mathrm{e-mg}} = V_\mathrm{u} - \frac{1}{2} \Re\mathrm{e}\, \mathcal{D}^M \vartheta_M{}^A \mathcal{P}_A = V_\mathrm{u} + \frac{1}{2} \mathcal{M}^{MN} \vartheta_M{}^A \vartheta_N{}^A \mathcal{P}_A \mathcal{P}_B\,, \tag{3.114}$$

where $\mathcal{M}$ is the symplectic matrix defined in eq. (3.15). It satisfies

$$\partial V_{\mathrm{e-mg}} / \partial \vartheta_M{}^A = -\Re\mathrm{e}(\mathcal{D}^M \mathcal{P}_A)\,. \tag{3.115}$$

There may exist a supermultiplet containing the 3-forms $C_A{}^M$ such that the supersymmetry algebra closes without the need to use a duality relation. We leave it to future work to study its possible (non-)existence.

### 3.3.6 The 3-form $C$ and the dual of the superpotential

We have seen that the consistency of the closure of the local supersymmetry algebra on the 2-forms $B_{\underline{\mathrm{a}}}$ and $B_\sharp$ requires the existence of a 3-form field that we have denoted by $C$, whose gauge transformation cancels the Stückelberg shift of those 2-forms.

An Ansatz for the supersymmetry transformation of $C$ can be made by writing down 3-form spinor bilinears that have zero Kähler weight and that are consistent with the chirality of the fermionic fields. Further, from eq. (3.71) it follows that there will be no gauge potential terms needed in the Ansatz. We thus make the following Ansatz

$$\delta_\epsilon C_{\mu\nu\rho} = -3i\eta \mathcal{L}\, \bar\epsilon^* \gamma_{[\mu\nu} \psi^*_{\rho]} - \frac{1}{2} \eta \mathcal{D}_i \mathcal{L} \bar\epsilon^* \gamma_{\mu\nu\rho} \chi^i + \mathrm{c.c.}\,, \tag{3.116}$$

where $\eta$ is a constant to be found. It turns out that the local supersymmetry algebra closes for two different reality conditions for $\eta$, which leads to the existence of two different 3-forms that we will call $C$ and $C'$.



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1. For $\eta = -i$ the algebra closes into the gauge transformations required by the 2-forms $B_{\underline{a}}$ and $B_\sharp$ provided that the field strength $G = dC$ vanishes. As discussed earlier there may be non-vanishing contributions if we were to construct the supersymmetry algebra at the quartic fermion order.

2. For $\eta \in \mathbb{R}$ the algebra closes into the following gauge transformation

$$\delta_{\text{gauge}} C' = -d\Lambda' \,, \tag{3.117}$$

where the 2-form $\Lambda'$ is given by

$$\Lambda' = c' - 2\eta \Im\text{m}(\mathcal{L}\phi) \,, \qquad c'_{\mu\nu} \equiv C'_{\mu\nu\rho}\xi^\rho \,, \tag{3.118}$$

provided the field strength $G' = dC'$ satisfies the duality relation

$$G' = \star\eta(-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*) \,. \tag{3.119}$$

Observe that the right hand side is nothing but the part of the scalar potential eq. (3.114) that depends on the superpotential. Actually, if we rescale the superpotential by $\mathcal{L} \to \eta\mathcal{L}$, then we can rewrite the above duality relation in the standard fashion

$$G' = \frac{1}{2} \star \frac{\partial V_{\text{e-mg}}}{\partial \eta} \,, \tag{3.120}$$

and, therefore, we can see the 3-form $C'$ as the dual of the deformation parameter associated to the superpotential, just as we can see the 3-forms $C_A{}^M$ as the duals of the deformation parameters $\vartheta_M{}^A$.

Observe that, had we chosen to work with a vanishing superpotential we would have found the duality rule $G' = 0$. This suggests a possible interpretation of the 3-form $C$ to be explored: that it may be related to another, as yet unknown, deformation of $N = 1, d = 4$ supergravity which has not been used. The full supersymmetric action is needed to confirm this possibility or to find, perhaps, a term bilinear in fermions which is dual to $C$.

Finally, observe that neither of the 3-forms $C, C'$ was predicted by the standard tensor hierarchy. $C$, though, is predicted by the extension associated to the constraints eqs. (3.30) and (3.107).

### 3.3.7 The 4-form fields $D_E{}^{NP}, D_{AB}, D^{NPQ}, D^M$

In the previous sections we have introduced four 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}, D^M$ in order to close the local supersymmetry algebra and have fully gauge-covariant field strengths. We thus expect that we can also find consistent supersymmetry transformations for all these 4-forms.

For the three 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$ there is a slight complication that has to do with the existence of extra Stückelberg shift symmetries. There are two such shift symmetries and in appendix C they correspond to the parameters $\tilde{\Lambda}_E{}^{(NP)}$ and $\Lambda_{BE}{}^P$. The origin of these symmetries lies in the fact that the $W$ tensors that appear in the field



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strengths of the 3-forms are not all independent. The symmetries result from the identities (B.16) and (B.17) together with the constraints $L_{NPQ} = Q^{AB} = Q_{NM}{}^{A} = 0$. This means that if we want to realize $N = 1$ supersymmetry on the 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}$ the parameters $\tilde{\Lambda}_E{}^{(NP)}$ and $\Lambda_{BE}{}^{P}$ will appear on the right hand side of commutators as part of the local algebra.

Most of these features are already visible in the simpler case of the ungauged theory,[21] i.e. for $\vartheta_M{}^{A} = 0$ and even when the ungauged case has no symmetries that act on the vectors, i.e. when all the matrices $T_A = 0$. We will restrict ourselves to realizing the supersymmetry algebra on the 4-forms for the ungauged theory with $T_A = 0$ for all $A$ for simplicity. The 4-form supersymmetry transformations in this simple setting are given by

$$\delta_\epsilon D_{AB} = -\frac{i}{2} \star \mathcal{P}_{[A} \partial_i \mathcal{P}_{B]} \bar{\epsilon} \chi^i + \text{c.c.} - B_{[A} \wedge \delta_\epsilon B_{B]} \,, \tag{3.121}$$

$$\delta_\epsilon D^{NPQ} = 10 A^{(N} \wedge F^P \wedge \delta_\epsilon A^{Q)} \,, \tag{3.122}$$

$$\delta_\epsilon D_E{}^{NP} = C_E{}^{P} \wedge \delta_\epsilon A^N \,. \tag{3.123}$$

$$\delta_\epsilon D^M = -\frac{i}{2} \star \mathcal{L}^* \bar{\epsilon} \lambda^M + \text{c.c.} + C \wedge \delta_\epsilon A^M \,. \tag{3.124}$$

When $\vartheta_M{}^{A} = 0$ and $T_A = 0$ the only place where there still appears a Stückelberg shift parameter is in the gauge transformation of $D_E{}^{NP}$. From the commutators we find that

$$\tilde{\Lambda}_E{}^{(NP)} = -2\Lambda^{(N} F^{P)} \wedge B_E \,. \tag{3.125}$$

### 3.4 The gauge-invariant bosonic action

It turns out that in order to write an action for the bosonic fields of the theory with electric and magnetic gaugings of perturbative and non-perturbative symmetries it is enough to add to the fundamental (electric) fields just the magnetic 1-forms $A_\Lambda$ and the 2-forms $B_A$. The gauge-invariant action takes the form

$$\begin{aligned} S_{\text{e-mg}} = \int \Big\{ & \star R - 2\mathcal{G}_{ij^*} \mathfrak{D} Z^i \wedge \star \mathfrak{D} Z^{*j^*} - 2\Im\text{m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re\text{e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma \\ & - \star V_{\text{e-mg}} - 4 Z^{\Sigma A} B_A \wedge \left( F_\Sigma - \frac{1}{2} Z_\Sigma{}^{B} B_B \right) - \frac{4}{3} X_{[MN]\Sigma} A^M \wedge A^N \wedge \left( F^\Sigma - Z^{\Sigma B} B_B \right) \\ & - \frac{2}{3} X_{[MN]}{}^{\Sigma} A^M \wedge A^N \wedge \left( dA_\Sigma - \frac{1}{4} X_{[PQ]\Sigma} A^P \wedge A^Q \right) \Big\} \,. \end{aligned} \tag{3.126}$$

The scalar potential $V_{\text{e-mg}}$ is given by eq. (3.114). Furthermore, the gauge transformations that leave invariant the above action ($\delta_a$) are those of the extended hierarchy ($\delta'_h$) except for the 2-forms:

$$\delta_a B_A = \delta'_h B_A - 2 T_{A\,NP} \Lambda^N \left( F^P - G^P \right) \,. \tag{3.127}$$

The action contains the 2-forms $B_A$ always contracted with $Z^{MA}$ so that we do not need to worry about the different behavior of $B_{\mathbf{a}}$ and $B_{\underline{a}}, B_\sharp$ under gauge transformation due to the extra constraint eq. (3.77).

[21] Note that the hierarchy remains non-trivial for $\vartheta_M{}^{A} = 0$.



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A general variation of the above action gives

$$\delta S = \int \left\{ \delta g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} + \left( \delta Z^i \frac{\delta S}{\delta Z^i} + \text{c.c.} \right) - \delta A^M \wedge \star \frac{\delta S}{\delta A^M} + 2\delta B_A \wedge \star \frac{\delta S}{\delta B_A} \right\} , \tag{3.128}$$

where the first variations with respect to the different fields are given by

$$\begin{aligned} -\star\frac{\delta S}{\delta g^{\mu\nu}} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*} \left[ \mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*j^*} - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*j^*} \right] \\ &\quad - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho} + \frac{1}{2} g_{\mu\nu} V_{\text{e-mg}} , \end{aligned} \tag{3.129}$$

$$-\frac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*} \mathfrak{D} \star \mathfrak{D} Z^{*j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\frac{1}{2}\partial_i V_{\text{e-mg}} , \tag{3.130}$$

$$-\frac{1}{4}\star\frac{\delta S}{\delta A^M} = \mathfrak{D} G_M - \frac{1}{4}\vartheta_M{}^A \star j_A + \frac{1}{2} T_{AMN} A^N \wedge \vartheta^{PA} (F_P - G_P) , \tag{3.131}$$

$$\star\frac{\delta S}{\delta B_A} = \vartheta^{PA} (F_P - G_P) . \tag{3.132}$$

The above equations are formally symplectic-covariant and, therefore, electric-magnetic duality symmetric. Both the Maxwell equations and the "Bianchi identities" have now sources to which they couple with a strength determined by the embedding tensor's electric and magnetic components.

It is expected to be possible to find a gauge-invariant action in which all the hierarchy's fields appear (as was done in [15]) if one assumes that none of the constraints on the embedding tensor are satisfied. Then, the 3-forms $C_A{}^M$ and the 4-forms $D_E{}^{NP}, D_{AB}, D^{NPQ}, D^M$ are introduced as Lagrange multipliers enforcing the constancy of the embedding tensor and the algebraic constraints $Q_{NP}{}^E = 0, Q^{AB} = 0$, $L_{NPQ} = 0$ and $(\vartheta_M{}^{\underline{a}} \mathcal{P}_{\underline{a}} + \vartheta_M{}^\sharp \mathcal{P}_\sharp)\mathcal{L} = 0$, respectively, but we will not study this possibility here.

It should be stressed that, even though the action eq. (3.126) contains $2n_V$ vectors and some number $n_B$ of 2-forms $B_a$ it does not carry all those degrees of freedom. To make manifest the actual number of degrees of freedom we briefly repeat here the arguments of [8] regarding the gauge fixing of the action (3.126). First, we choose a basis of magnetic vectors and generators such that the non-zero entries of $\vartheta^{\Lambda a}$ arrange themselves into a square invertible submatrix $\vartheta^{Ii}$. We split accordingly $A_{\Lambda\mu} = (A_{I\mu}, A_{U\mu})$. It can be shown by looking at the vector equations of motion that the Lagrangian does not depend on the $A_{U\mu}$, i.e. $\delta\mathcal{L}/\delta A_{U\mu} = 0$. Further, the electric vectors $A^I{}_\mu$ that are dual to the magnetic vectors $A_{I\mu}$, which are used in some gauging, have massive gauge transformations, $\delta A^I{}_\mu = -\mathfrak{D}_\mu \Lambda^I - \vartheta^{Ii}\Lambda_{i\mu}$ and can be gauged away. The $n_B$ 2-forms $B_i$ can by eliminated from the Lagrangian by using their equations of motion eq. (3.132). The 2-forms appear without derivatives in eq. (3.132) so that it is possible to solve for them and to substitute the on-shell expression back into the action. This is allowed as the 2-forms appear everywhere (up to partial integrations) without derivatives. One then ends up with an action depending on $n_B$ magnetic vectors $A_{I\mu}$ and $n_V - n_B$ electric vectors $A^U{}_\mu$.

The relation between the tensor hierarchy and the action (or its equations of motion) as well as the physical interpretation of the field content of the extended hierarchy will be discussed in the next section.



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## 4 Summary and conclusions

We have discussed the possible symmetries of $N = 1, d = 4$ supergravity and their gauging using as gauge fields both electric and magnetic vectors.

When using both electric and magnetic 1-forms as gauge fields at the same time one is also compelled to introduce 2-forms $B_A$, associated to all the possible symmetries of the theory. For each electric vector $A^\Lambda$ whose magnetic dual $A_\Lambda$ is gauged, because the magnetic components of the embedding tensor $\vartheta^{\Lambda A}$ do not vanish, one introduces a 2-form $\vartheta^{\Lambda A} B_A$ in its field strength. $A^\Lambda$ has a massive gauge transformation and it forms a Stückelberg pair with the 2-form $\vartheta^{\Lambda A} B_A$. By electro-magnetic duality we end up with Stückelberg pairs $A^M, \vartheta_M{}^A B_A$.

The embedding tensor-projected 2-forms $\vartheta_M{}^{\mathbf{a}} B_{\mathbf{a}}$ are dual to the embedding tensor-projected Noether currents that are associated to gauged isometry directions $\vartheta_M{}^{\mathbf{a}} j_{\mathbf{a}}$ whereas the remaining 2-forms $B_{\mathbf{a}}$ are dual to ungauged isometry directions. The 2-forms $B_{\underline{a}}$ and $B_\sharp$ are pure gauge at lowest order in fermions, but it is to be expected that they are actually dual to the Noether currents associated to the respective symmetries, which are bilinear in fermions. To properly test this idea one would have to construct the supersymmetry algebra at quartic order in fermions.

We have seen that the presence of a non-vanishing superpotential breaks the global symmetries that we have denoted with the indices ¿ $\underline{\mathrm{a}}, \#$. Thus, if $\mathcal{L} \neq 0$, we must set $(\vartheta_M{}^{\underline{\mathrm{a}}} \mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^\sharp \mathcal{P}_\sharp) = 0$, which is a new constraint that the embedding tensor must satisfy. We have written it in the form eq. (3.30) to handle the cases $\mathcal{L} = 0$ and $\mathcal{L} \neq 0$ simultaneously. When $\mathcal{L} \neq 0$, then, $N = 1, d = 4$ supersymmetry implies that the 2-forms $B_{\underline{\mathrm{a}}}, B_\sharp$ transform under new Stückelberg shifts parametrized by a 2-form gauge transformation parameter $\Lambda$. Still, since $\Lambda \neq 0$ only when $\mathcal{L} \neq 0$, and in this case we have to impose the new constraint (something we have expressed through eq. (3.77)), the gauge transformations of the projected 2-forms $Z^{MA} B_A$ are left unchanged by the new 2-form Stückelberg shifts. Therefore the field strengths $F^M$ and the action keep their standard form.

In the standard tensor hierarchy it is necessary to introduce 3-forms $C_A{}^M$ to construct gauge-covariant field strengths $H_A$ for the 2-forms $B_A$. These 3-forms are the dual of the embedding tensor $\vartheta_M{}^A$. However, when $\mathcal{L} \neq 0$, the standard tensor hierarchy field strengths $H_A$ need to be modified by the addition of a 3-form $C$, into $H'_A$, see eq. (3.80). The 3-form $C$ must absorb the new Stückelberg shifts of the 2-forms $B_{\underline{\mathrm{a}}}, B_\sharp$, but one has to show that $N = 1, d = 4$ supergravity allows for such a 3-form.

We have found consistent supersymmetry transformation rules for two 3-forms $C$ and $C'$ the first of which has precisely the required gauge transformations. $C'$ is unexpected from the hierarchy point of view but turns out to be the dual of the superpotential, seen as a deformation of the ungauged theory. The fact that it is not predicted by the hierarchy (even in its extended form which includes the constraint eq. (3.30)) is due to the fact that the superpotential is not associated to any gauge symmetry, which is the basis of the tensor hierarchy. On the other hand, the existence of the 3-form $C$ suggests the possible existence of another deformation of $N = 1, d = 4$ supergravity unrelated to gauge symmetry and to the superpotential.



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Again, in the $\mathcal{L} \neq 0$ case the field strengths ${G_C}^M$ need to be modified by the addition of new 4-forms $D^M$ not predicted by the standard hierarchy, which must absorb gauge transformations related to $\Lambda$. In the standard hierarchy the 4-forms ${D_E}^{NP}, D_{AB}, D^{NPQ}$ are associated to the constraints ${Q_{NP}}^E, Q^{AB}, L_{NPQ}$. The fourth 4-form that appears when $\mathcal{L} \neq 0$ in $N = 1, d = 4$ supergravity could well be related to the constraint $(\vartheta_M{}^{\underline{\mathrm{a}}}\mathcal{P}_{\underline{\mathrm{a}}} + \vartheta_M{}^{\sharp}\mathcal{P}_{\sharp}) = 0$ that the embedding tensor must satisfy. This can only be fully confirmed by the construction of a supersymmetric action containing all the $p$-forms as in [15]. Nevertheless, it is clear that, when we vary the action without any constraints imposed on the embedding tensor, we expect it to be necessary to introduce a 4-form $D^M$ multiplying that constraint. The gauge transformations of the 4-forms $D^M$ should compensate for this lack of gauge invariance.

Some, but not all, of the $p$-forms in the hierarchy may be associated to dynamical supersymmetric branes. In order to construct a $\kappa$-symmetric action for a $(p-1)$-brane that couples to a certain $p$-form, two necessary conditions are that the $p$-form transforms under no Stückelberg shift and that under supersymmetry transform into a gravitino multiplied by some scalars may couple to branes. In $N = 1, d = 4$ supergravity the $p$-forms that satisfy this condition are the (subset) of 2-forms $B_{\mathbf{a}}$ whose gauge transformations are massless. These are the 2-forms whose field strengths are dual to ungauged isometry currents. From the analysis of [19, 27] we know that these couple to strings (one-branes that have been referred to as stringy cosmic strings). Another form which satisfies the criteria is the 3-form $C'$ which is a natural candidate to describe couplings to domain walls. We note that there are no 1-forms and 4-forms that can couple to a massive brane. There are thus no 1/2 BPS black holes in the theory and no 1/2 BPS space-time filling branes. The latter fact may be qualitatively understood from the fact that one cannot truncate the minimal $N = 1, d = 4$ supersymmetry algebra to a supersymmetry algebra with half of the original supercharges.


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## A Kähler geometry

A Kähler manifold is a complex manifold on which there exist complex coordinates $Z^i$ and $Z^{*\,i^*} = (Z^i)^*$ and a real function $\mathcal{K}(Z, Z^*)$, called the *Kähler potential*, such that the

$$ds^2 = 2\mathcal{G}_{ii^*}\, dZ^i dZ^{*\,i^*}\,, \tag{A.1}$$



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with

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*} \mathcal{K} \,. \tag{A.2}$$

The *Kähler (connection) 1-form* $\mathcal{Q}$ is defined by

$$\mathcal{Q} \equiv \frac{1}{2i}(dZ^i \partial_i \mathcal{K} - dZ^{*\,i^*} \partial_{i^*} \mathcal{K}) \,, \tag{A.3}$$

and the *Kähler 2-form* $\mathcal{J}$ is its exterior derivative

$$\mathcal{J} \equiv d\mathcal{Q} = i\mathcal{G}_{ii^*} dZ^i \wedge dZ^{*\,i^*} \,. \tag{A.4}$$

The Kähler potential is defined only up to *Kähler transformations*

$$\mathcal{K}'(Z, Z^*) = \mathcal{K}(Z, Z^*) + f(Z) + f^*(Z^*) \,, \tag{A.5}$$

where $f(Z)$ is any holomorphic function of the complex coordinates $Z^i$ that leave the Kähler metric and 2-form invariant. The components of the Kähler connection 1-form transform according to

$$\mathcal{Q}'_i = \mathcal{Q}_i - \frac{i}{2}\partial_i f \,. \tag{A.6}$$

Objects with Kähler weight $(q, \bar{q})$ transform by definition under the above Kähler transformations with a factor $e^{-(qf+\bar{q}f^*)/2}$ and their Kähler-covariant derivative $\mathcal{D}$ is

$$\mathcal{D}_i \equiv \nabla_i + iq\mathcal{Q}_i \,, \qquad \mathcal{D}_{i^*} \equiv \nabla_{i^*} - i\bar{q}\mathcal{Q}_{i^*} \,, \tag{A.7}$$

where $\nabla$ is the standard covariant derivative associated to the Levi-Cività connection. The Ricci identity for this covariant derivative is, on objects without any indices and Kähler weight $(q, \bar{q})$

$$[\mathcal{D}_i, \mathcal{D}_{j^*}] = \frac{1}{2}(\bar{q} - q)\mathcal{G}_{ij^*} \,. \tag{A.8}$$

When $(q, \bar{q}) = (1, -1)$, this defines a complex line bundle over the Kähler manifold whose first, and only, Chern class equals the Kähler 2-form $\mathcal{J}$, i.e. a *Kähler-Hodge (KH) manifold*. These are the manifolds parametrized by the complex scalars of the chiral multiplets of $N = 1, d = 4$ supergravity. Furthermore, objects such as the superpotential and all the spinors of the theory have a well-defined Kähler weight.

We will often use the spacetime pullback of the Kähler-covariant derivative on tensor fields with Kähler weight $(q, -q)$ (weight $q$, for short):

$$\mathfrak{D}_\mu = \nabla_\mu + iq\mathcal{Q}_\mu \,, \tag{A.9}$$

where $\nabla_\mu$ is the standard spacetime (and/or Lorentz-) covariant derivative plus possibly the pullback of the Levi-Cività connection. $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form

$$\mathcal{Q}_\mu = \frac{1}{2i}(\partial_\mu Z^i \partial_i \mathcal{K} - \partial_\mu Z^{*\,i^*} \partial_{i^*} \mathcal{K}) \,. \tag{A.10}$$



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## B Projectors of the $d = 4$ tensor hierarchy

The 4-dimensional hierarchy's field strengths are defined in terms of the invariant tensors $Z^{MA}, Y_{AM}{}^{B}, W_C{}^{MAB}, W_{CNPQ}{}^{M}, W_{CNP}{}^{EM}$ which act as projectors. In this appendix we collect their definitions and the properties that they satisfy.

The projectors are defined by

$$Z^{PA} \equiv -\frac{1}{2}\Omega^{NP}\vartheta_N{}^{A} = \begin{cases} +\frac{1}{2}\vartheta^{\Lambda A}\,, \\ \\ -\frac{1}{2}\vartheta_{\Lambda}{}^{A}\,, \end{cases} \tag{B.1}$$

$$Y_{AM}{}^{C} \equiv \vartheta_M{}^{B} f_{AB}{}^{C} - T_{A\,M}{}^{N}\vartheta_N{}^{C}\,, \tag{B.2}$$

$$W_C{}^{MAB} \equiv -Z^{M[A}\delta_C{}^{B]}\,, \tag{B.3}$$

$$W_{CNPQ}{}^{M} \equiv T_{C\,(NP}\delta_{Q)}{}^{M}\,, \tag{B.4}$$

$$W_{CNP}{}^{EM} \equiv \vartheta_N{}^{D} f_{CD}{}^{E}\delta_P{}^{M} + X_{NP}{}^{M}\delta_C{}^{E} - Y_{CP}{}^{E}\delta_N{}^{M}\,. \tag{B.5}$$

They satisfy the orthogonality relations

$$Z^{MA}Y_{AN}{}^{C} = \frac{1}{2}\Omega^{PM}Q_{PN}{}^{C} = 0\,, \tag{B.6}$$

$$Y_{AM}{}^{C}W_C{}^{MAB} = Y_{AM}{}^{C}W_{CNPQ}{}^{M} = Y_{AM}{}^{C}W_{CNP}{}^{EM} = 0\,. \tag{B.7}$$

The $W$ projectors are related to the embedding tensor constraints by

$$\vartheta_M{}^{C}W_C{}^{MAB} = 2Q^{AB}\,, \tag{B.8}$$

$$\vartheta_M{}^{C}W_{CNPQ}{}^{M} = L_{NPQ}\,, \tag{B.9}$$

$$\vartheta_M{}^{C}W_{CNP}{}^{EM} = 2Q_{NP}{}^{E}\,. \tag{B.10}$$

Under variations we have

$$\delta\vartheta_M{}^{C}W_C{}^{MAB} = \vartheta_M{}^{C}\delta W_C{}^{MAB} = \frac{1}{2}\delta(\vartheta_M{}^{C}W_C{}^{MAB}) = \delta Q^{AB}\,, \tag{B.11}$$

$$\delta\vartheta_M{}^{C}W_{CNPQ}{}^{M} = \delta L_{NPQ}\,, \tag{B.12}$$

$$\delta\vartheta_M{}^{C}W_{CNP}{}^{EM} = \vartheta_M{}^{C}\delta W_{CNP}{}^{EM} = \frac{1}{2}\delta(\vartheta_M{}^{C}W_{CNP}{}^{EM}) = \delta Q_{NP}{}^{E}\,. \tag{B.13}$$

The constraints eqs. (3.22), (3.25) and (3.26) are related through the following identities

$$Q^{AB}Y_{BP}{}^{E} - \frac{1}{2}Z^{NA}Q_{NP}{}^{E} = 0\,, \tag{B.14}$$

$$Q_{(MN)}{}^{A} - 3L_{MNP}Z^{PA} - 2Q^{AB}T_{BMN} = 0\,, \tag{B.15}$$

where eq. (B.14) can be obtained from eq. (B.15) by multiplying the latter by $Z^{NE}$. Differentiating these identities with respect to the embedding tensor, using eqs. (B.11)–(B.13), we also find the following relations among the $W$ tensors:

$$W_C{}^{MAB}Y_{BP}{}^{E} - \frac{1}{2}Z^{NA}W_{CNP}{}^{EM} - \frac{1}{4}Q^{M}{}_{P}{}^{E}\delta_C^A + Q^{AB}\left[\delta_P^M f_{BC}{}^{E} - T_{BP}{}^{M}\delta_C^E\right] = 0\,, \tag{B.16}$$

$$W_{C(MN)}{}^{AQ} - 3W_{CMNP}{}^{Q}Z^{PA} - \frac{3}{2}L_{MN}{}^{Q}\delta_C{}^{A} - 2W_C{}^{QAB}T_{B\,MN} = 0\,. \tag{B.17}$$



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## C Gauge transformations and field strengths of the $d = 4$ tensor hierarchy

The gauge transformations of the different fields of the tensor hierarchy are

$$\delta_h A^M = -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,, \tag{C.1}$$

$$\delta_h B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}\left[\Lambda^N F^P + \frac{1}{2}A^N \wedge \delta_h A^P\right] - Y_{AM}{}^C \Lambda_C{}^M\,, \tag{C.2}$$

$$\begin{aligned}\delta_h C_C{}^M = {}& \mathfrak{D}\Lambda_C{}^M - F^M \wedge \Lambda_C - \delta_h A^M \wedge B_C - \frac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta_h A^P \\ & + \Lambda^M H_C - W_C{}^{MAB}\Lambda_{AB} - W_{CNPQ}{}^M \Lambda^{NPQ} - W_{CNP}{}^{EM}\Lambda_E{}^{NP}\,,\end{aligned} \tag{C.3}$$

$$\begin{aligned}\delta_h D_{AB} = {}& \mathfrak{D}\Lambda_{AB} + 2T_{[AMN}\tilde{\Lambda}_{B]}{}^{(MN)} + Y_{[A|P}{}^E(\Lambda_{B]E}{}^P - B_{B]} \wedge \Lambda_E{}^P) + \mathfrak{D}\Lambda_{[A} \wedge B_{B]} \\ & - 2\Lambda_{[A} \wedge H_{B]} + 2T_{[A|NP}\left[\Lambda^N F^P - \frac{1}{2}A^N \wedge \delta_h A^P\right] \wedge B_{|B]}\,,\end{aligned} \tag{C.4}$$

$$\begin{aligned}\delta_h D_E{}^{NP} = {}& \mathfrak{D}\Lambda_E{}^{NP} + \tilde{\Lambda}_E{}^{(NP)} + \frac{1}{2}Z^{NB}\Lambda_{BE}{}^P - F^N \wedge \Lambda_E{}^P \\ & + C_E{}^P \wedge \delta_h A^N + \frac{1}{12}T_{EQR}A^N \wedge A^P \wedge A^Q \wedge \delta_h A^R + \Lambda^N G_E{}^P\,,\end{aligned} \tag{C.5}$$

$$\begin{aligned}\delta_h D^{NPQ} = {}& \mathfrak{D}\Lambda^{NPQ} - 3Z^{(N|A}\tilde{\Lambda}_A{}^{|PQ)} - 2A^{(N} \wedge dA^P \wedge \delta_h A^{Q)} \\ & - \frac{3}{4}X_{RS}{}^{(N}A^{P|} \wedge A^R \wedge A^S \wedge \delta_h A^{|Q)} - 3\Lambda^{(N}F^P \wedge F^{Q)}\,,\end{aligned} \tag{C.6}$$

where we remark that $\Lambda_E{}^{NP}$ is a 3-form and $\tilde{\Lambda}_E{}^{(NP)}$ is a 4-form.

Their gauge-covariant field strengths are

$$F^M = dA^M + \frac{1}{2}X_{[NP]}{}^M A^N \wedge A^P + Z^{MA}B_A\,, \tag{C.7}$$

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge \left[dA^S + \frac{1}{3}X_{NP}{}^S A^N \wedge A^P\right] + Y_{AM}{}^C C_C{}^M\,, \tag{C.8}$$

$$\begin{aligned}G_C{}^M = {}& \mathfrak{D}C_C{}^M + \left[F^M - \frac{1}{2}Z^{MA}B_A\right] \wedge B_C + \frac{1}{3}T_{C\,SQ}A^M \wedge A^S \wedge dA^Q \\ & + \frac{1}{12}T_{C\,SQ}X_{NT}{}^Q A^M \wedge A^S \wedge A^N \wedge A^T \\ & + W_C{}^{MAB}D_{AB} + W_{CNPQ}{}^M D^{NPQ} + W_{CNP}{}^{EM}D_E{}^{NP}\,.\end{aligned} \tag{C.9}$$

These field strengths are related by the following hierarchical Bianchi identities

$$\mathfrak{D}F^M = Z^{MA}H_A\,, \tag{C.10}$$

$$\mathfrak{D}H_A = Y_{AM}{}^C G_C{}^M + T_{A\,MN}F^M \wedge F^N\,. \tag{C.11}$$

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